Wave Inversion Technology, Report No. 1, pages 227-234

2D seismic modeling in transversely isotropic media with a Chebyshev-Fourier method in consideration of the free surface and the surface/grid interface topography

T. Ruedas, E. Teßmer[1]

keywords: *seismic forward modeling, spectral methods, topography, transverse isotropy*

ABSTRACT

The fundamentals of a program for seismic forward modeling in 2D transversely isotropic media based on a spectral Chebyshev-Fourier method are described. Especially, the boundary conditions for the free surface, the bottom of the numerical grid and, in case of grid coupling, the interface between an acoustic (isotropic) and a transversely isotropic medium have to be taken into account; in a realistic modeling, the topography of the free surface resp. the interface between coupled grids should be considered.

INTRODUCTION

A possible approach for seismic modeling is the direct solution of the equations of dynamic elasticity. One of several categories of solution methods is formed by the spectral methods, where spatial derivatives are calculated by some kind of transformation. In Chebyshev-Fourier modeling (Kosloff et al., 1990), the horizontal derivatives are calculated by the Fourier transform, the vertical derivatives by certain theorems of the Chebyshev approximations of the respective functions. The definition of boundary conditions is essential to numerical stability in the Chebyshev method and can be done by introducing the physical boundary conditions for the upper and lower grid boundaries; for their implementation, characteristic variables (Gottlieb et al., 1982) have been used. In the horizontal direction, the boundary conditions are periodical due to the Fourier method. — In both cases, efficiency is improved by the use of FFTs.

Stability considerations (Teßmer, 1990) demand the use the velocity-stress formulation of the equations of dynamic elasticity (Virieux, 1986), which is of first order in time. For time-stepping, a fourth-order Taylor operator is used.

[1]**email:** ekkehart.tessmer@dkrz.de

THE NUMERICAL GRID

For the modeling discussed here, a special numerical grid is used. Because of the use of the Fourier method, the grid columns are equidistant; in contrast to this, the positions of the N base points for the Chebyshev approximation of a function are given by the extrema of the $(N-1)$th Chebyshev polynomial, which lie much more dense near the interval margins than in its center. Stability and efficiency considerations give cause for a nonlinear stretching of the original Chebyshev grid, so that the grid point intervals at the grid margins are increased (Kosloff and Tal-Ezer, 1993). In this way, a rectangular "auxiliary" grid for the numerical algorithm is generated; it has the coordinates ξ and ζ.

To incorporate the topography, the rectangular grid is deformed by a topography function, which stretches the grid columnwise in the vertical direction in accordance with the shape of the topography; the simplest mapping is to stretch the grid in such a way that the deformation of the grid rows is largest at the surface and decreases linearly down to the bottom (Tessmer et al., 1992b). In the case of coupled grids, the shape of the interface topography controls the grid deformation. However, the geological model for the modeling run is to be generated in the "physical grid" with the coordinates $x(\xi,\zeta)$ and $z(\xi,\zeta)$ created by such a coordinate transformation. — The topography must be differentiable and should be periodical because of the Fourier method.

In spectral methods, very coarse grids are used, which is an advantage from the point of view of efficiency and memory. On the other hand, the representation of the subsurface geometry is not as accurate as in finer grids; in general, a smooth line representing an interface between two media does not have a smooth, but a stairlike shape in a grid. In coarse grids, the stairs can be so large, that the short-wavelength parts of the wave field are strongly diffracted at interfaces, causing much noise in seismograms and snapshots. This effect can be reduced by a certain method of averaging the material parameters in grid cells crossed by interfaces (Muir et al., 1992), where the materials are substituted by an anisotropic medium.

TWO-DIMENSIONAL EQUATIONS OF MOTION

One grid — transversely isotropic (elastic) medium

In the velocity-stress formulation of the equations of dynamic elasticity, a system of five coupled differential equations has to be solved in the case of a transversely isotropic medium with symmetry axis of arbitrary orientation in the x-z-plane:

$$
\begin{aligned}
\varrho\frac{\partial \dot u_x}{\partial t} &= \frac{\partial \sigma_{xx}}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial \sigma_{xz}}{\partial \zeta}\frac{\partial \zeta}{\partial z} + f_x \\
\varrho\frac{\partial \dot u_z}{\partial t} &= \frac{\partial \sigma_{xz}}{\partial \xi} + \frac{\partial \sigma_{xz}}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial \sigma_{zz}}{\partial \zeta}\frac{\partial \zeta}{\partial z} + f_z \qquad (1)\\
\frac{\partial \sigma_{xx}}{\partial t} &= c_{11}\left(\frac{\partial \dot u_x}{\partial \xi} + \frac{\partial \dot u_x}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) + c_{13}\frac{\partial \dot u_z}{\partial \zeta}\frac{\partial \zeta}{\partial z} + c_{15}\left(\frac{\partial \dot u_z}{\partial \xi} + \frac{\partial \dot u_z}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial \dot u_x}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right)
\end{aligned}
$$

$$\frac{\partial \sigma_{zz}}{\partial t} = c_{13}\left(\frac{\partial \dot{u}_x}{\partial \xi} + \frac{\partial \dot{u}_x}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) + c_{33}\frac{\partial \dot{u}_z}{\partial \zeta}\frac{\partial \zeta}{\partial z} + c_{35}\left(\frac{\partial \dot{u}_z}{\partial \xi} + \frac{\partial \dot{u}_z}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial \dot{u}_x}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right)$$
$$\frac{\partial \sigma_{xz}}{\partial t} = c_{15}\left(\frac{\partial \dot{u}_x}{\partial \xi} + \frac{\partial \dot{u}_x}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right) + c_{35}\frac{\partial \dot{u}_z}{\partial \zeta}\frac{\partial \zeta}{\partial z} + c_{55}\left(\frac{\partial \dot{u}_z}{\partial \xi} + \frac{\partial \dot{u}_z}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial \dot{u}_x}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right) ;$$

here $\dot{u}_x$ and $\dot{u}_z$ are the components of the particle velocity, σ_{xx}, σ_{zz} and σ_{xz} are the components of the stress tensor, ϱ is the density of the material and the c_{ij} are the elements of its elasticity matrix (using Voigt notation). The terms $\frac{\partial \zeta}{\partial x}$ and $\frac{\partial \zeta}{\partial z}$ are due to the chain rule and link the physical to the auxiliary grid.

After solving the system of equations, the boundary conditions for the free surface at the grid top and for the bottom have to be applied. Using characteristic variables, which describe wave propagation normal to the boundary, certain correction terms have to be added to the original particle velocity and stress values at the boundary before applying the boundary conditions. For the free surface, the boundary conditions are $\sigma_{zz} = \sigma_{xz} = 0$; for the bottom boundary, a paraxial transparent boundary can be implemented to suppress at least strong near vertical reflections (see Appendix A). Additionally, a damping boundary strip is necessary at the model bottom as well as at the sides (Cerjan et al., 1985), where the periodicity of the Fourier transform makes waves leaving the model at one side enter it at the other.

Two coupled grids — acoustic/elastic medium

There are two reasons for coupling an acoustic and an elastic grid, e.g. in marine models: first, the grids can have different spacings, which may increase the efficiency of the method; second, the smoothing mentioned in Section cannot be applied at acoustic/elastic interfaces, so that the diffractions from the interface stairs will produce lots of noise. In the elastic lower grid of such a marine model, the eqs. (1) are valid; in the acoustic upper part, the system

$$\varrho\frac{\partial \dot{u}_x}{\partial t} = \frac{\partial \sigma_a}{\partial \xi} + \frac{\partial \sigma_a}{\partial \zeta}\frac{\partial \zeta}{\partial x} + f_x$$
$$\varrho\frac{\partial \dot{u}_z}{\partial t} = \frac{\partial \sigma_a}{\partial \zeta}\frac{\partial \zeta}{\partial z} + f_z \qquad (2)$$
$$\frac{\partial \sigma_a}{\partial t} = \lambda\left(\frac{\partial \dot{u}_x}{\partial \xi} + \frac{\partial \dot{u}_x}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial \dot{u}_z}{\partial \zeta}\frac{\partial \zeta}{\partial z}\right)$$

after (Tessmer et al., 1992a) with the bulk modulus $\lambda = \varrho v_P^2 = c_{13}$ and the pressure $p = -\sigma_a$ must be solved. — The boundary conditions are applied in the same way as for a single grid. The coupling is done by assigning the same particle velocities and stresses to the grid points at the interface boundary of each grid after correction with the characteristic variables (see Appendix A).

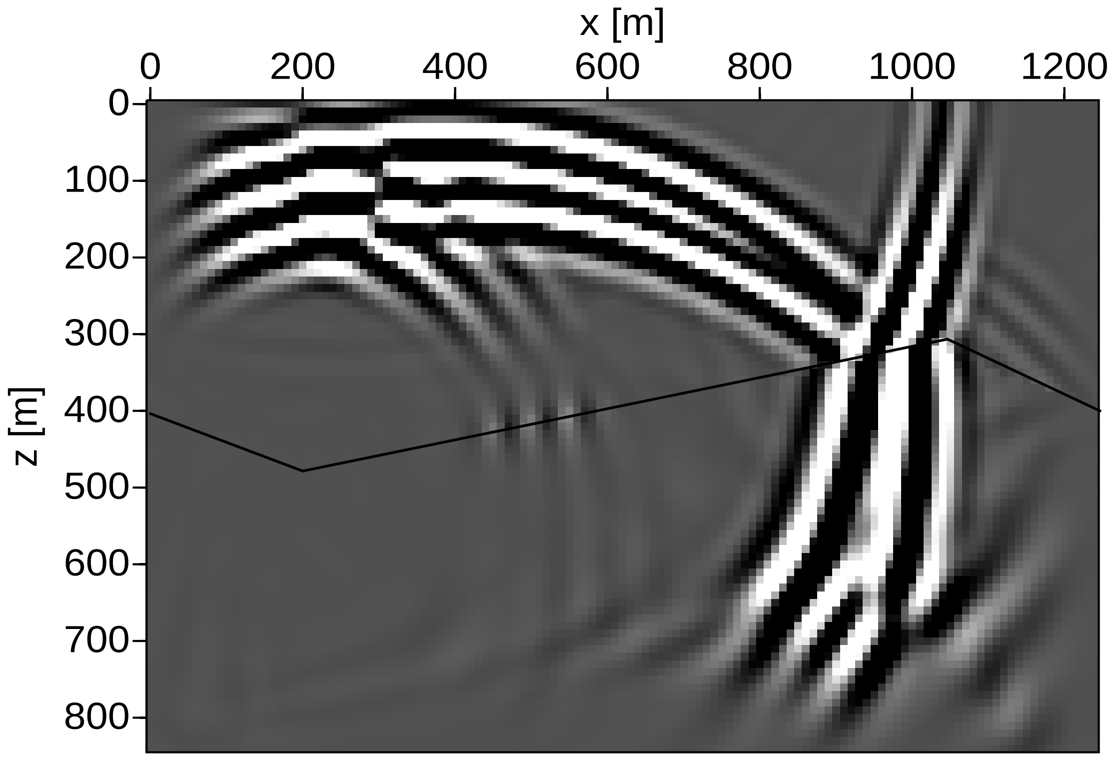

Figure 1: Acoustic/elastic model, consisting of sea water $(\varrho = 1.02\,\frac{\mathrm{g}}{\mathrm{cm}^3}, \lambda = 2.295\,\mathrm{GPa})$ and a transversely isotropic solid $(\varrho = 2.075\,\frac{\mathrm{g}}{\mathrm{cm}^3}$, $c_{11} = 31.257\,\mathrm{GPa}$, $c_{13} = 3.399\,\mathrm{GPa}$, $c_{33} = 22.487\,\mathrm{GPa}$, $c_{55} = 6.486\,\mathrm{GPa}$; $\alpha = 10°)$; the figure shows $\dot{u}_z$ at $t = 0.59865$ s. The time history of the explosion source $(\oslash = 20$ m$)$ at $(240\,\mathrm{m}, 41.59\,\mathrm{m})$ is given by a Ricker wavelet with a cutoff frequency of 50 Hz.

CONCLUSION

The Chebyshev-Fourier method allows a comprehensive simulation of the propagation of elastic waves in models including isotropic as well as anisotropic media and reaches a high degree of accuracy, as can be shown by comparison with analytical solutions. The incorporation of the topography is essential for the modeling of realistic geological models, e.g. in the interpretation of field data; especially the treatment of marine models with this method is improved considerably by the possibility of coupling two (or more) grids.

REFERENCES

Bayliss, A., Jordan, K. E., LeMesurier, B. J., and Turkel, E., 1986, A fourth-order accurate finite-difference scheme for the computation of elastic waves: Bull.Seis.Soc.Am., **76**, 1115–1132.

Cerjan, C., Kosloff, D., Kosloff, R., and Reshef, M., 1985, A nonreflecting boundary condition for the discrete acoustic and elastic wave equations: Geophysics, **50**, 705–708.

Gottlieb, D., Gunzburger, M., and Turkel, E., 1982, On numerical boundary treatment of hyperbolic systems for finite difference and finite element methods: SIAM J.Num.Anal., **19**, 671–682.

Kosloff, D., and Tal-Ezer, H., 1993, A Modified Chebyshev pseudospectral method with an $O(N^{-1})$ time step restriction: J.Comp.Phys., **104**, 457–469.

Kosloff, D., Kessler, D., Filho, A. Q., Tessmer, E., Behle, A., and Strahilevitz, R., 1990, Solution of the equations of dynamic elasticity by a Chebychev spectral method: Geophysics, **55**, 734–748.

Muir, F., Dellinger, J., Etgen, J., and Nichols, D., September 1992, Modeling elastic fields across irregular boundaries: Geophysics, **57**, no. 9, 1189–1193.

Ruedas, T., 1997, Entwicklung eines Programms zur zweidimensionalen seismischen Modellierung für transversalisotrope Medien mit einer Tschebyscheff-Fourier-Methode unter Einbeziehung der freien Oberfläche mit Topographie: Master's thesis, Universität Hamburg.

Tessmer, E., Kessler, D., Kosloff, D., and Behle, A., 1992a, Multi-domain Chebyshev-Fourier method for the solution of the equations of motion of dynamic elasticity: J.Comp.Phys., **100**, 355–363.

Tessmer, E., Kosloff, D., and Behle, A., 1992b, Elastic wave propagation simulation in the presence of surface topography: Geophys.J.Int., **108**, 621–632.

Teßmer, M. E., 1990, Seismische Modellierung unter Berücksichtigung der freien Oberfläche mithilfe einer spektralen Tschebyscheff-Methode: Ph.D. thesis, Universität Hamburg.

Virieux, J., 1986, P-SV wave propagation in heterogeneous media: Velocity-stress finite-difference method: Geophysics, **51**, 888–901.

APPENDIX A

In this appendix, the corrections with the characteristic variables are listed. In the formulae, the following abbreviations are used:

$$\beta_{1,2} = \sqrt{\frac{1}{2\varrho}\left(c_{33} + c_{55} \pm \sqrt{(c_{33} - c_{55})^2 + 4c_{35}^2}\right)}$$

are the qP and the qSV phase velocity in the z direction;

$$\gamma_1 := \frac{c_{35}}{\beta_1^2\varrho - c_{55}} \qquad \gamma_2 := \frac{c_{35}}{\beta_2^2\varrho - c_{33}}$$
$$\delta_u := 1 - \gamma_1\gamma_2, \qquad \delta_\sigma := c_{33}c_{55} - c_{35}^2;$$

$$a = \tfrac{\beta_1}{\delta_\sigma}(\gamma_2 c_{33} + c_{35}) \qquad b = \tfrac{\beta_1}{\delta_\sigma}(\gamma_2 c_{35} + c_{55})$$
$$c = \tfrac{\beta_2}{\delta_\sigma}(c_{33} + \gamma_1 c_{35}) \qquad d = \tfrac{\beta_2}{\delta_\sigma}(c_{35} + \gamma_1 c_{55}).$$

In the following, the corrected variables are marked with a tilde. They are valid for a model, where the normal of the boundary coincides with the z axis, which of course is not the case in the presence of topography; then all variables and moduli which are not invariant to rotation have to be rotated into a local coordinate system with the local z axis normal to the local boundary before the correction, and back afterwards ((Tessmer et al., 1992b)). For the free surface of a transversely isotropic medium, the boundary conditions $\tilde{\sigma}_{zz} = \tilde{\sigma}_{xz} = 0$ lead to

$$\tilde{\dot{u}}_x = \dot{u}_x + \sigma_{xz}\frac{c - \gamma_1 a}{\delta_u} + \sigma_{zz}\frac{\gamma_1 b - d}{\delta_u} \tag{1a}$$

$$\tilde{\dot{u}}_z = \dot{u}_z + \sigma_{xz}\frac{\gamma_2 c - a}{\delta_u} + \sigma_{zz}\frac{b - \gamma_2 d}{\delta_u} \tag{1b}$$

$$\tilde{\sigma}_{xx} = \sigma_{xx} - \frac{\sigma_{xz}}{\delta_\sigma}(c_{15}c_{33} - c_{13}c_{35}) - \frac{\sigma_{zz}}{\delta_\sigma}(c_{13}c_{55} - c_{15}c_{35}); \tag{1c}$$

similarly, the free surface condition $\tilde{\sigma}_a = 0$ for an acoustic medium yields

$$\tilde{\dot{u}}_z = \dot{u}_z + \frac{\sigma_a}{\sqrt{\varrho\lambda}}. \tag{2}$$

The following corrections represent a paraxial transparent boundary condition for the bottom of a grid with a transversely isotropic medium:

$$\tilde{\dot{u}}_x = \frac{1}{2}\left(\dot{u}_x - \sigma_{xz}\frac{c - \gamma_1 a}{\delta_u} - \sigma_{zz}\frac{\gamma_1 b - d}{\delta_u}\right) \tag{3a}$$

$$\tilde{\dot{u}}_z = \frac{1}{2}\left(\dot{u}_z - \sigma_{xz}\frac{\gamma_2 c - a}{\delta_u} - \sigma_{zz}\frac{b - \gamma_2 d}{\delta_u}\right) \tag{3b}$$

$$\tilde{\sigma}_{xz} = \frac{1}{2}\left(\sigma_{xz} - \dot{u}_x\frac{b - \gamma_2 d}{bc - ad} - \dot{u}_z\frac{d - \gamma_1 b}{bc - ad}\right) \tag{3c}$$

$$\tilde{\sigma}_{zz} = \frac{1}{2}\left(\sigma_{zz} - \dot{u}_x\frac{a - \gamma_2 c}{bc - ad} - \dot{u}_z\frac{c - \gamma_1 a}{bc - ad}\right) \tag{3d}$$

$$\tilde{\sigma}_{xx} = \sigma_{xx} + \frac{\tilde{\sigma}_{xz} - \sigma_{xz}}{\delta_\sigma}(c_{15}c_{33} - c_{13}c_{35}) + \frac{\tilde{\sigma}_{zz} - \sigma_{zz}}{\delta_\sigma}(c_{13}c_{55} - c_{15}c_{35}). \tag{3e}$$

If an acoustic, isotropic upper grid is coupled with an elastic, transversely isotropic lower grid, the waves leaving the upper grid downward and the waves leaving the lower grid upward shall be conserved at the interface. The continuity conditions $\tilde{\dot{u}}_{z_a} = \tilde{\dot{u}}_{z_e} =: \tilde{\dot{u}}_z$, $\tilde{\sigma}_a = \tilde{\sigma}_{zz_e} =: \tilde{\sigma}_{zz}$ and $\tilde{\sigma}_{xz_e} = 0$ lead to

$$\begin{aligned}
\tilde{\sigma}_{zz} &= \left(\frac{1}{\sqrt{\varrho_a c_{13_a}}} + \frac{b - \gamma_2 d}{\delta_u} \right)^{-1} \cdot \\
&\qquad \left(\dot{u}_{z_e} - \dot{u}_{z_a} + \sigma_a \frac{1}{\sqrt{\varrho_a c_{13_a}}} + \sigma_{zz_e} \frac{b - \gamma_2 d}{\delta_u} + \sigma_{xz_e} \frac{\gamma_2 c - a}{\delta_u} \right) && \text{(4a)} \\
\tilde{\dot{u}}_z &= \dot{u}_{z_e} + (\sigma_{zz_e} - \tilde{\sigma}_{zz}) \frac{b - \gamma_2 d}{\delta_u} + \sigma_{xz_e} \frac{\gamma_2 c - a}{\delta_u} && \text{(4b)} \\
\tilde{\dot{u}}_{x_e} &= \dot{u}_{x_e} + (\sigma_{zz_e} - \tilde{\sigma}_{zz}) \frac{\gamma_1 b - d}{\delta_u} + \sigma_{xz_e} \frac{c - \gamma_1 a}{\delta_u} && \text{(4c)} \\
\tilde{\sigma}_{xx_e} &= \sigma_{xx_e} - (\sigma_{zz_e} - \tilde{\sigma}_{zz}) \frac{c_{13} c_{55} - c_{15} c_{35}}{\delta_\sigma} - \sigma_{xz_e} \frac{c_{15} c_{33} - c_{13} c_{35}}{\delta_\sigma}. && \text{(4d)}
\end{aligned}$$