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A 3-D FD Eikonal Solver for Non-Cubical Grids

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ABSTRACT

Conventional finite-difference (FD) eikonal solvers work on a cubical grid, i.e., a grid with identical spacing along each axis. This fact renders these methods inflexible with respect to the grid spacing of the input velocity model. In addition, a carefully chosen set of grid spacings can reduce the model size and thus save computational time, plus it may even allow for stronger velocity contrasts to be handled without the appearance of acausalities. We present a fast and accurate FD eikonal solver that operates on a non-cubical grid. Consequently, it overcomes the limitations with respect to the spacing of the input velocity model. Also, it can use the non-cubical grid to save computational time by reducing the number of grid points and can handle stronger velocity contrasts in certain situations.

INTRODUCTION

Efficient calculation of first arrival travel times in gridded 3-D volumes plays an important role in various areas of applied geophysics such as tomography, earth quake location and also prestack Kirchhoff depth migration. The class of methods most widely known and commonly accepted because of the combination of excellent speed and good accuracy still is the FD eikonal solvers as originally developed by Vidale (1988, 1990). This is especially true for the 3-D case considered here because of the sheer amount of computations that must be carried out to fill a 3-D volume with travel times at each sampling point.

As already mentioned, conventional 3-D FD eikonal solvers operate on cubical grids, i.e., rectangular grids, where the grid spacing is the same for each coordinate axis.

The use of a cubical grid implies unwanted restrictions of the sampling of the input velocity model. Often, the original model data is not equally spaced and thus has to be resampled. But non-cubical grids offer other advantages, too. Depending on the

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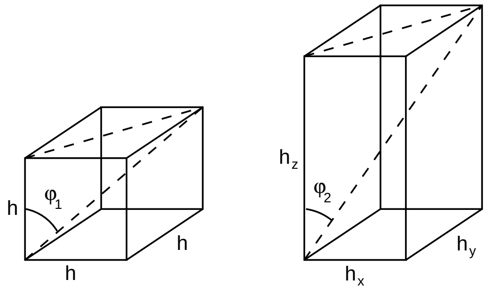

Figure 1: Minimum critical angles for cubical (left) and non-cubical (right) grids.

variation of the model one can resample it, making the grid as fine as necessary but as coarse as possible to the model correctly on the one hand and minimize computational time on the other.

Finally, by stretching the grid in the appropriate direction one can extend the maximum velocity contrast that can be treated without acausalities for this direction. This is due to the fact that the minimum critical angle of incidence depends on the geometry of the grid cell. For a cubical cell one gets (compare figure 1, left):

$$\tan\varphi_1 = \sqrt{\frac{2}{3}} \quad \text{or} \quad \varphi_1 \approx 39.23°, \tag{1}$$

whereas when e.g. $h_y = h_x$ and $h_z = 2h_x$ the minimum critical angle becomes (compare figure 1, right):

$$\tan\varphi_2 = \sqrt{\frac{2}{6}} \quad \text{or} \quad \varphi_1 = 30°. \tag{2}$$

That means that in the first case a velocity contrast from, e. g., 2000 m/s to 2449 m/s can be handled whereas this rises to 2000 m/s to 3464 m/s for the second case. When $h_y = h_x$ and $h_z = 4h_x$ it even becomes 2000 m/s to 6000 m/s.

In this paper, we present a new implementation that allows for the use of non-cubical grids, i.e., grids that can have different grid spacings along different axes. This makes it possible to use all itemized advantages of non-cubical grids. The application to two analytically solvable models demonstrates speed and accuracy of the algorithm.

METHODOLOGY

The formulae for the non-cubical scheme are not directly derived from the eikonal equation by replacing the partial derivatives with finite difference expressions. They are rather constructed by finding appropriate values for weights w_0 to w_6 in a general formula by introducing certain constraints. As an example, the general formula for the by far most frequent situation, where seven time samples t_0 to t_6 at the corners of a grid box are known and the eighth t_7 has to be computed (for the geometry see figure 2):

$$\begin{aligned} t_7 &= t_0 + \left(w_0 - w_1 \frac{(t_1 - t_2)^2 + (t_5 - t_6)^2}{2} \right. \\ &\quad - w_2 \frac{(t_2 - t_4)^2 + (t_3 - t_5)^2}{2} - w_3 \frac{(t_4 - t_1)^2 + (t_6 - t_3)^2}{2} \\ &\quad - w_4 \frac{(t_3 - t_2)^2 + (t_5 - t_4)^2}{2} - w_5 \frac{(t_3 - t_1)^2 + (t_6 - t_4)^2}{2} \\ &\quad \left. - w_6 \frac{(t_5 - t_1)^2 + (t_6 - t_2)^2}{2} \right)^{\frac{1}{2}} \end{aligned} \tag{3}$$

To obtain correct and stable results also for fairly large differences between the grid spacings, we switch from a model representation that uses different grid spacings h_x, h_y and h_z and an isotropic velocity v, to a formulation with a set of anisotropic velocities v_x, v_y and v_z and a unique grid spacing h. With these velocities as given below we use phase velocity rather than group velocity to compute travel times which makes sure that travel times are computed in a correct way. The anisotropic velocities along the axes are:

$$v_x = v\frac{h_x}{h}, \quad v_y = v\tfrac{h_y}{h}, \quad v_z = v\frac{h_z}{h}. \tag{4}$$

Accordingly, diagonal and big diagonal velocities read:

$$\begin{aligned} v_{xy} &= \sqrt{\frac{v_x^2 + v_y^2}{2}} \\ v_{yz} &= \sqrt{\frac{v_x^2 + v_z^2}{2}} \\ v_{yz} &= \sqrt{\frac{v_y^2 + v_z^2}{2}} \\ v_{xyz} &= \sqrt{\frac{v_x^2 + v_y^2 + v_z^2}{3}}. \end{aligned} \tag{5}$$

With this formulation of the non-cubical grid as anisotropic velocities one can find the following weights for the above travel time formula:

$$\begin{aligned} w_0 &= \frac{3h^2}{v_{xyz}} \\ w_1 &= w_2 = w_3 = 1 \\ w_4 &= \frac{6h^2 + (-2v_{xy}^2 - 2v_{yz}^2 + v_y^2 + v_z^2)w_0}{4h^2} \end{aligned} \tag{6}$$

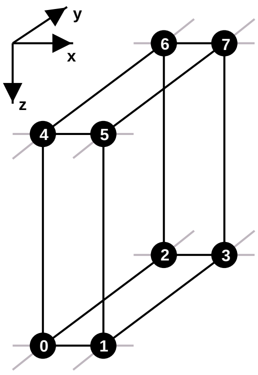

Figure 2: Cell of the scheme of computation.

$$
\begin{aligned}
w_5 &= \frac{6h^2 + (v_x^2 - 2v_{xy}^2 - 2v_{yz}^2 + v_z^2)w_0}{4h^2} \\
w_6 &= \frac{6h^2 + (v_x^2 - 2v_{xz}^2 + v_y^2 - 2v_{yz}^2)w_0}{4h^2}
\end{aligned}
$$

The algorithm itself remains almost unchanged from the original by Vidale (1990): After an initialization around the source, the scheme of computation proceeds at the surface of an expanding box. First sides and then edges are computed subsequently and independently for each surface, until the whole volume is filled with travel times.

APPLICATION

We apply our non-cubical FD eikonal solver to two 3-D models with identical dimensions but different velocity distributions, a constant gradient and a two layers case.

The models are box-shaped with edge lengths $x = 500$ m, $y = 3000$ m and $z = 2000$ m in the respective directions. The corresponding grid spacings are $h_x = 10$ m, $h_y = 20$ m and $h_z = 40$ m giving a total number of $51 \times 151 \times 51 = 392751$ samples.

The computational time necessary for computing first arrival travel times for each of the 392751 subsurface points is for both models about 1.75 s. This time was measured with the utility *gprof* on a Sun Workstation with an Ultra2 processor running at 250 MHz. The code was compiled with maximum optimization and ran on one single processor. The required time is about 80 % longer than for the same number of grid points and a cubical-grid. However, with an equal grid spacing of 10 m in each direction, the number of grid points would be about ten fold of the current number resulting in a computational time five times the one for the non-cubical grid.

The given errors are absolute values of relative errors between the numerically computed and the analytically for the gradient model or semi-analytically for the two layers model travel times.

Constant Gradient Model

The gradient model's top velocity is $v_t = 2000$ m/s, the gradient is $g = 1.5$ 1/s yielding a bottom velocity $v_b = 5000$ m/s.

The time slices in figure 5 for the gradient model show a very good fit between numerically computed and true travel times. This is also reflected by the small average relative error $err_{av} = 0.0024$ %. The maximum relative error $err_{max} = 2.22$ % appears in the top left backpart of the model due to small angles of incidence of the expanding wavefront relative to the model grid. These small angles can numerically not be treated very well with the FD formulae.

Two Layers Model

The upper layer velocity of the two layer model is $v_u = 4000$ m/s, the lower layer velocity $v_l = 7000$ m/s.

Figure 6 shows that also for the two layer model accuracy of the method is excellent. The average relative error is with $err_{av} = 0.0004$ % even half an order of magnitude smaller than for the gradient model. Again the maximum relative error $err_{max} = 2.20$ % appears in the top left backpart of the model for the reasons given above. The interface region between the two layers as well as the head wave, however, show no significant error, but are treated correctly.

CONCLUSIONS AND OUTLOOK

We presented an efficient, i.e., fast and accurate, FD eikonal solver that does not operate on the usual cubical but on a non-cubical grid. This allows for higher flexibility with regard to the sampling of the input model and, more important, can save substantial amounts of core memory because the model sampling can be adapted to the model structure in such a way that least resources are used. Another advantage can be the ability to handle stronger velocity contrasts without the occurrence of acausalities as was shown in the second numerical example.

The non-cubical method is currently slower than the cubical-one. However, we hope to be able to diminish the difference by better scooping the optimization potential of the non-cubical method.

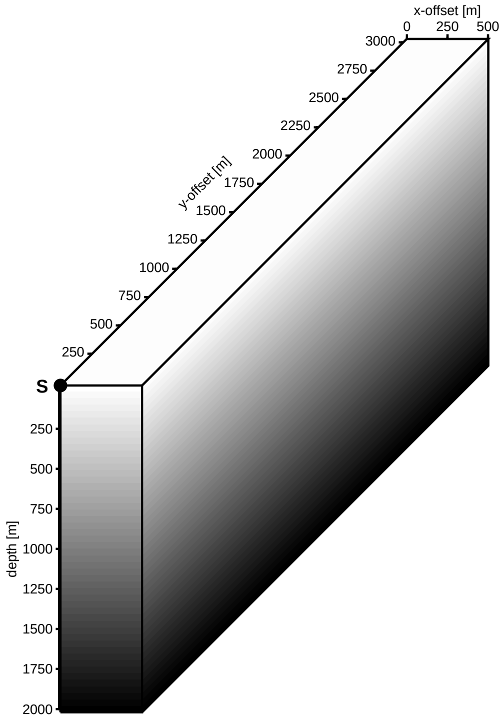

Figure 3: Constant velocity gradient: Darker color denotes higher velocity.

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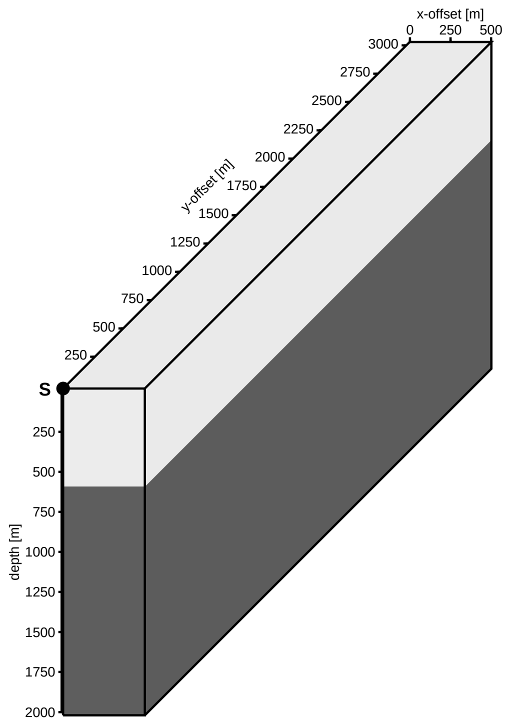

Figure 4: Two layer model: Darker color denotes higher velocity.

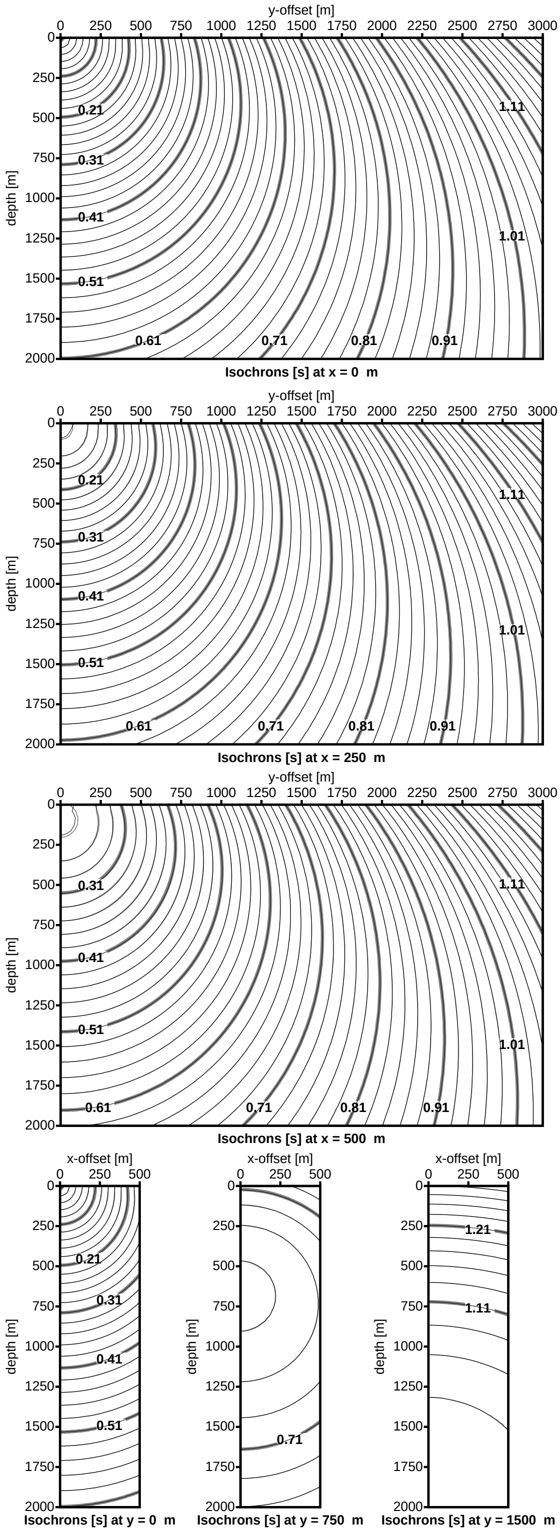

Figure 5: Constant velocity gradient: Thin lines denote numerically computed travel times, thick grey lines analytic ones.

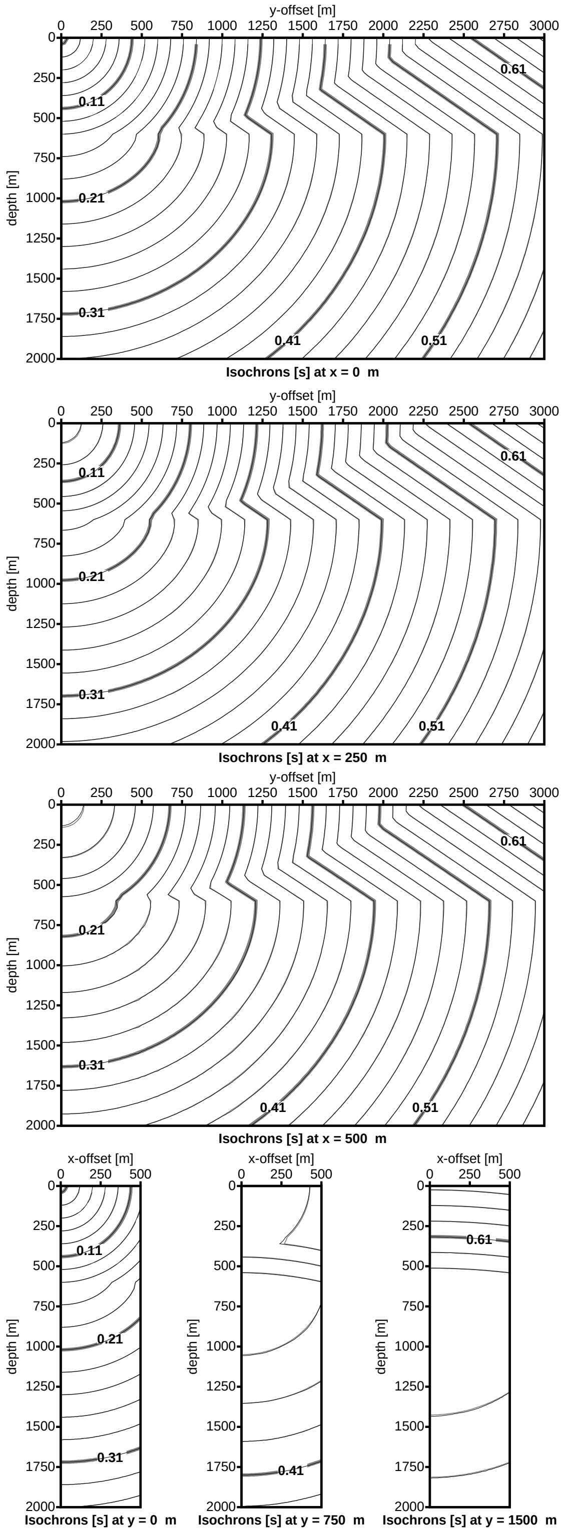

Figure 6: Two layer model: Thin lines denote numerically computed travel times, thick grey lines analytic ones.