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What may happen if the velocities of qS waves in an anisotropic inhomogeneous medium coincide at a point ?

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ABSTRACT

Time domain equations for anisotropic inhomogeneous media form the complicated hyperbolic system of partial differential equations, which possess some peculiarities unknown in the elastodynamics of isotropic media. One of them appears when the velocities of qS waves coincide at a point on a ray. It turns out that the corresponding mathematical problem is rather complicated in the general case, so we study such a phenomenon on a simplified model problem where rigorous mathematical treatment is available. The main results are the following. The ray ansatz fails in a vicinity of the point where the velocities of two possible waves coincide. The wave field is described there by a more complicated mathematical expression. One wave, being incident at this point, gives rise to both waves behind the point.

Ray tracing in anisotropic inhomogeneous media sometimes faces the situation when the ray displays a directional jump. Thus it happens at a point where the velocities of qS waves coincide.

Let us dwell on the mathematical peculiarities of such points. If we deal with the ray ansatz for the displacement vector $\boldsymbol{U}(\boldsymbol{x}, t)$:

$$\begin{aligned} \boldsymbol{U}(\boldsymbol{x}, t) &= \sum_{k=0}^{\infty} \boldsymbol{u}_k(\boldsymbol{x}) F_k(t - \tau(\boldsymbol{x})) \\ \frac{d}{d\xi} F_{k+1}(\xi) &= F_k(\xi) \ , \end{aligned} \tag{1}$$

then the following sequence of recurrent equations has to be resolved:

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$$\begin{aligned} N\boldsymbol{u}_0 &= 0 \\ N\boldsymbol{u}_1 + M\boldsymbol{u}_0 &= 0 \\ N\boldsymbol{u}_{k+2} + M\boldsymbol{u}_{k+1} + L\boldsymbol{u}_k &= 0\ , \qquad k = 0, 1, 2, ... \end{aligned} \tag{2}$$

where we use the notations commonly accepted in geophysical literature. Note, the first equation in (2) is the system of linear algebraic equations with a certain symmetrical 3×3 matrix $N = N(\tau, \boldsymbol{x})$, while both M and L are matrix differential operators. The eikonal equations come out from the condition $det N = 0$. They are closely related to the eigenvalues of N. The eigenvectors of N are called the polarization vectors.

Suppose now that the velocity of the qP wave is separated from the velocities of qS waves and never coincides with them. Then by means of a smooth linear transformation we can split the eigen-subspace of N corresponding to the qP wave and for the remaining two eigenvalues λ_1 and λ_2 we obtain the following analytical expressions:

$$\lambda_{1,2} = A \pm \sqrt{D}\ , \tag{3}$$

with A and D both being smooth functions of position $\boldsymbol{x}$. Thus, $\lambda_1 = \lambda_2$ at points where $D = 0$, but exactly at these points the partial derivatives of the eigenvalues become singular. Apparently, the corresponding eigenvectors, or the polarization vectors, will not be differentiable functions of the position, where $D = 0$ (see eq. (3)), and already the second equation in (2), where the transport equation comes from, becomes singular. Therefore the ray method technique faces serious problems already with the amplitude of the leading term $\boldsymbol{u}_0(\boldsymbol{x}, t)$. Clearly, no need to add that this technique cannot be employed for higher order terms.

From the mathematical point of view, the problem we face here for the general case of anisotropic inhomogeneous media is rather complicated. To develop a physical insight to the processes in a vicinity of the points under consideration, we study, from one side, a simplified model problem but, from the other side, we treat it in mathematically rigorous way. Therefore the results are reliable and do not contain any additional heuristic assumptions.

The model can be interpreted as two elastic inhomogeneous strings which are finite from one side, $x = 0$, infinite from the other side and interacting via a potential. The velocity of the first string (channel 1) and the second one (channel 2) are denoted by $a_1(x)$ and $a_2(x)$, respectively. At the end point, $x = 0$, one of the possible two waves is generated by a source at the time t=0 and we follow its propagation in space-time $x > 0$, $t > 0$.

If the velocities are different everywhere, i.e. $a_1(x) \neq a_2(x)$ for $x \geq 0$, then the ray ansatz (1) is valid everywhere for $x \geq 0$. The corresponding wave-train propagates with velocity $a_1(x)$ only. It is depicted schematically in Fig.1, left part, for the following set of functions F_k in (1):

$$\begin{aligned} F_0(\xi) &= \delta(\xi)\,, \qquad (\delta\ is\ the\ delta-function) \\ F_1(\xi) &= H(\xi)\,, \qquad (H\ is\ the\ Heaviside-step\ function) \\ F_2(\xi) &= \xi_+ \qquad etc. \end{aligned} \tag{4}$$

If $a_1 = a_2$, but $\frac{da_1}{dx} \neq \frac{da_2}{dx}$ at a point $x = x*$, then $\boldsymbol{u}_k(x*)$ starting with $k = 2, 3, ...$ become singular and we cannot use them in some vicinity of this point anymore. In this vicinity, we obtain a more complicated representation for the wave field via the sum of iterations to a certain system of integral equations (see middle part in Fig.1). But then, on some distance behind the point $x = x*$ we get, apart from the initial wave, a new wave propagating in channel 2 with velocity $a_2(x)$ and having specific discontinuities on its wave front

$$t = \tau_2(x) = \int_0^{x*} \frac{d\xi}{a_1(\xi)} + \int_{x*}^{x} \frac{d\xi}{a_2(\xi)}\,,$$

which cannot be described by the initial set of functions F_k (see eq.(4)). So-called derivatives of non-integer order appear behind the point $x = x*$. Using another terminology we can say that the shape of that new wave, or the wavelet, is different from what we have for the incident wave. The corresponding wave-picture is depicted schematically on the right part of Fig.1 for the case $a_2(x) > a_1(x)$ for $x > x*$.

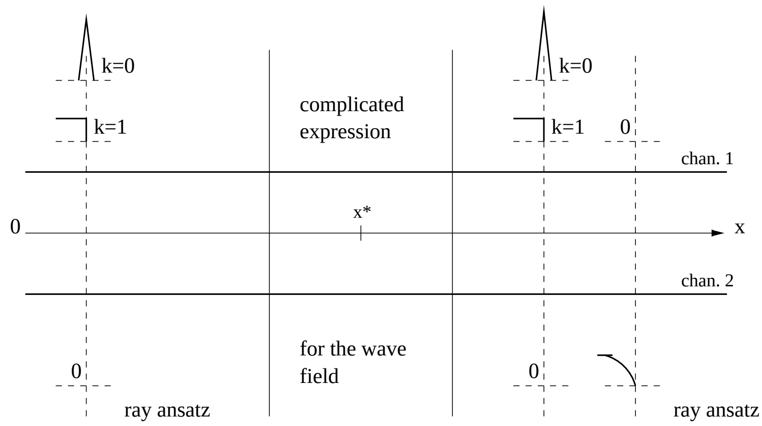

Figure 1:

Based on the results obtained for the simplified model problem we could anticipate that in the general case of anisotropic inhomogeneous media i) the ray ansatz fails in a vicinity of such points and it should be replaced by some mathematically more complicated formula, and ii) starting with only one incident qS wave before the point we will get both of them behind this point $x = x*$ and the second one may be of the same order as the initial wave.

In conclusion we would like to pay attention to an analogy between the problem of caustics, for instance, in isotropic inhomogeneous media and the problem of coinciding qS wave velocities in anisotropic inhomogeneous media. When we deal with a separate ray, both problems arise as local problems, singularities appear at a point and can stop numerical calculations. If, nevertheless, we jump over the point in the computational process, we will loose the phase-shift in the first problem and another qS wave in the second one. But at the same time both problems, in fact, are not local because the same difficulties will take place for the other rays from a ray tube around the initial one. Note, that the condition $D = 0$ describes a manifold in 3D and not only a point.

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