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The Kichhoff-Helmholtz Integral for Anisotropic Elastic Media

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ABSTRACT

A generalization of the Kirchhoff-Helmholtz approximation to anisotropic elastic media is achieved by replacing the unknown scattered field in the Kirchhoff integral representation by the specular reflected field at the interface. The latter is represented by the geometrical ray approximation of the incident field multiplied with the amplitude-normalized plane-wave reflection coefficient and with an exchanged polarization vector according to a specular reflection obeying Snell's law. The stationary-phase evaluation shows that the high-frequency result of the obtained integral approximation of the reflected field closely resembles the geometrical ray approximation. If the phase velocities at the scattering point appearing in the integral are replaced by the respective group velocities, the correspondence is exact.

INTRODUCTION

Wave propagation is often qualitatively described using Huygen's principle, which states that the superposition of secondary sources along the wavefront at a certain time produces the next wavefront at a later time. Using the divergence theorem, this principle can be quantified in the Kirchhoff integral (Sommerfeld, 1964). The wavefield at the observation point is calculated by an integral over the field and its derivative along a surface completely surrounding the observation point, provided the sources are located outside that surface.

For anisotropic media, with suitable boundary conditions on the Green's function, the reflected wavefield from a smooth interface can be expressed as a surface integral in terms of the upgoing wavefield and its normal derivative at the interface. This integral, which can be derived from Betti's theorem (see, e.g., Aki and Richards, 1980), is the generalized Kirchhoff integral.

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The Kirchhoff-Helmholtz approximation, which is a generalization of the physical-optics approximation (Bleistein, 1984) will be used in the Kirchhoff integral. The unknown upgoing wavefield is approximated by the known specularly-reflected wavefield from the source.

The geometric ray approximation (GRA) (Cervený, 1995, Chapman and Coates, 1994) will be used to approximate the Green's functions from the source to the interface and from the interface to the receiver. The Green's function can represent any type of wave, so that the results will be valid for multiply reflected and converted waves in anisotropic media, to the extent in which the GRA is valid. The final result is a generalization to anisotropic media of the standard Kirchhoff-Helmholtz integral (Tygel et al., 1994) and of the results of Fraser and Sen (1985) for isotropic media. In order to check the the amplitude response of the new integral, a stationary-phase analysis will be performed as in Ursin and Tygel (1998).

The new integral is non-reciprocal, so it would be interesting to compare it numerically with the reciprocal formula derived by Ursin and Tygel (1998) using the Born approximation.

THE KIRCHHOFF-HELMHOLTZ INTEGRAL

Wave propagation in an inhomogeneous anisotropic elastic solid, in the a absence of sources, is governed by the equations of motion (Aki and Richards, 1980)

$$\rho \ddot{u}_i - (c_{ijkl} u_{k,l})_{,j} = 0 \ , \tag{1}$$

where $u_i = u_i(\mathbf{x}, t)$ is the i-th component of the displacement vector $\mathbf{u}(\mathbf{x})$, $\rho = \rho(\mathbf{x})$ is the density and $c_{ijkl} = c_{ijkl}(\mathbf{x})$ are the elastic parameters of the medium at the point $\mathbf{x} = (x_1, x_2, x_3)$. The elastic parameters satisfy the symmetry relations $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. In equation (1) "$\dot{}$" stands for $\partial/\partial t$ and "$, j$" for $\partial/\partial x_j$. Also, a repeated index implies summation with respect to this index.

The equation of motion for the Green's function $g_{in}(\mathbf{x}, t; \mathbf{x}^s)$ is

$$\rho \ddot{g}_{in} - (c_{ijkl} g_{kn,l})_{,j} = \delta_{in} \delta(\mathbf{x} - \mathbf{x}^s) \delta(t) \ , \tag{2}$$

with initial conditions

$$g_{in}(\mathbf{x}, t; \mathbf{x}^s) = \dot{g}_{in}(\mathbf{x}, t; \mathbf{x}^s) = 0 \quad \text{for } t < 0 \quad \text{and } \mathbf{x} \neq \mathbf{x}^s \ . \tag{3}$$

The Green's function also satisfies the reciprocity relation (Aki and Richards, 1980, equation (2.39))

$$g_{in}(\mathbf{x}, t; \mathbf{x}^s) = g_{ni}(\mathbf{x}^s, t; \mathbf{x}) \ . \tag{4}$$

The (temporal) Fourier transform of the Green's function, $G_{in}(\mathbf{x}, \omega; \mathbf{x}^s)$, satisfies the elastic Helmholtz equation

$$-\omega^2 \rho G_{in} - (c_{ijkl} G_{kn,l})_{,j} = \delta_{in} \delta(\mathbf{x} - \mathbf{x}^s) \ . \tag{5}$$

The reciprocity relation (4) now becomes

$$\mathbf{G}(\mathbf{x}, \omega; \mathbf{x}^s) = \mathbf{G}^T(\mathbf{x}^s, \omega; \mathbf{x}) \ . \tag{6}$$

We shall use the geometric ray approximation (GRA) to obtain the approximate Green's function. Using the results in Cervený (1995), we have for a specific ray connecting a source point $\mathbf{x}^s$ to a scattering point $\mathbf{x}$, the GRA Green's function

$$G_{ij}(\mathbf{x}, \omega; \mathbf{x}^s) = h_i^s(\mathbf{x}) \ \frac{a(\mathbf{x}, \mathbf{x}^s) \ e^{i\omega T(\mathbf{x}, \mathbf{x}^s)}}{[\rho(\mathbf{x}) v^s(\mathbf{x}) \rho(\mathbf{x}^s) v(\mathbf{x}^s)]^{1/2}} \ h_j(\mathbf{x}^s) \ , \tag{7}$$

where $\mathbf{h}(\mathbf{x}^s)$ and $\mathbf{h}^s(\mathbf{x})$ are the unit polarization vectors, $\rho(\mathbf{x}^s)$ and $\rho(\mathbf{x})$ are the densities and $v(\mathbf{x}^s)$ and $v^s(\mathbf{x})$ the phase velocities in ray direction at the source $\mathbf{x}^s$ and at the point $\mathbf{x}$, respectively. Moreover, $T(\mathbf{x}, \mathbf{x}^s)$ is the traveltime along the ray from $\mathbf{x}$ to $\mathbf{x}^s$ and

$$a(\mathbf{x}, \mathbf{x}^s) = \frac{e^{-i\frac{\pi}{2} \operatorname{sgn}(\omega) \, \sigma(\mathbf{x}, \mathbf{x}^s)}}{4\pi |\det \underset{\sim}{\boldsymbol{Q}}_2(\mathbf{x}, \mathbf{x}^s)|^{1/2}} \tag{8}$$

is a complex amplitude function taking into account possible caustics and phase-shift at the source. In this expression, $|\det \underset{\sim}{\boldsymbol{Q}}_2(\mathbf{x}, \mathbf{x}^s)|^{1/2}$ denotes the relative geometric spreading factor and $\sigma(\mathbf{x}, \mathbf{x}^s)$ is the KMAH index for the ray that connects the source $\mathbf{x}^s$ to the point $\mathbf{x}$. We shall approximate the spatial derivatives by

$$\begin{aligned} G_{ij,k}(\mathbf{x}, \omega; \mathbf{x}^s) &\approx i\omega \, T_{,k}(\mathbf{x}, \mathbf{x}^s) \, G_{ij}(\mathbf{x}, \omega; \mathbf{x}^s) \\ &= i\omega \, p_k^s G_{ij}(\mathbf{x}, \omega; \mathbf{x}^s) \end{aligned} \tag{9}$$

where $p_k^s = p_k^s(\mathbf{x})$ is the kth component of the slowness vector $\mathbf{p}^s(\mathbf{x})$ at the point $\mathbf{x}$ (for the ray from the source).

We shall consider the wavefield from a source at $\mathbf{x}^s$ that is being reflected from a surface Σ and recorded at the point $\mathbf{x}^r$, as shown in Figure 1. The wavefield at $\mathbf{x}^r$ can be expressed as a surface integral involving the displacement field $\mathbf{u}(\mathbf{x})$ at the surface Σ by using a representation theorem that is given in Aki and Richards (1980), equation (2.41). In the absence of body forces, and with a Green's function that satisfies the reciprocity relation (4), this gives

$$\begin{aligned} u_m(\mathbf{x}^r, \omega) &= \int_\Sigma \Big\{ G_{mi}(\mathbf{x}^r, \omega, \mathbf{x}) \, c_{ijkl}(\mathbf{x}) \, u_{k,l}(\mathbf{x}, \omega) \\ &- G_{mk,l}(\mathbf{x}^r, \omega, \mathbf{x}) \, c_{ijkl}(\mathbf{x}) \, u_i(\mathbf{x}, \omega) \Big\} \mathbf{n}_j \, d\sigma \ . \end{aligned} \tag{10}$$

We want to find an expression for the Green's function for the reflected field from Σ due to a point sorce at x^s. In order to this, we need an approximation for the upgoing field at Σ which can then be substituted into the surface integral above. We propose to use the generalized Kirchhoff approximation (see Bleistein, 1984)

$$G_{ij}^{\text{ref}}(\mathbf{x},\omega,x^s) = h_i^{\text{spec}}(\mathbf{x})\,\frac{R(\mathbf{x},\mathbf{p}^s)\,a(\mathbf{x},\mathbf{x}^s)\;e^{i\omega T(\mathbf{x},\mathbf{x}^s)}}{[\rho(\mathbf{x})\,v^s(\mathbf{x})\,\rho(\mathbf{x}^s)\,v(\mathbf{x}^s)]^{1/2}}\;h_j(\mathbf{x}^s)\;, \qquad (11)$$

where $\mathbf{h}^{\text{spec}}$ is the polarization vector corresponding to a specular reflected wave of proper type at the point x on Σ, due to the wave from x^s and $R(\mathrm{x},\mathbf{p}^s)$ is the plane-wave reflection coefficient (normalized with respect to displacement amplitude) for our choice of incoming and outgoing type of wave.

The approximation of the derivative of this Green's function at the point x is given by

$$G_{ij,k}^{\text{ref}}(\mathbf{x},\omega,\mathbf{x}^s) \approx i\omega p_k^{\text{spec}} G_{ij}^{\text{ref}}(\mathbf{x}^r,\omega,\mathbf{x}^s)\;, \qquad (12)$$

where $\mathbf{p}^{\text{spec}}$ is the slowness vector of the specular reflected wave. The integral in equation (10) now yields for the recorded field at x^r

$$\begin{aligned}\Delta G_{mn}(\mathbf{x}^r,\omega,\mathbf{x}^s) \;=\; & \int_\Sigma \Big\{ G_{mi}(\mathbf{x}^r,\omega,\mathbf{x})\,c_{ijkl}(\mathbf{x})\,G_{kn,l}^{\text{ref}}(\mathbf{x},\omega,\mathbf{x}^s) \\ & - \; G_{mk,l}(\mathbf{x}^r,\omega,\mathbf{x})\,c_{ijkl}(\mathbf{x})\,G_{in}^{\text{ref}}(\mathbf{x},\omega)\Big\}\mathbf{n}_j\,d\sigma\,. \end{aligned} \qquad (13)$$

Here, we shall use the GRA Green's function from the source at x^s to the point x for a specified wave (given by a specific ray code) as represented by equation (7). We shall also use the corresponding GRA Green's function from the point x to the receiver at x^r for another specified wave (also given by a specified ray code). A possible wave-mode conversion at x is taken care of by selecting the proper reflection coefficient in equation (11). With the Kirchhoff approximations (11) and (12), equation (13) can be approximated by

$$\begin{aligned}\Delta G_{mn}(\mathbf{x}^r,\omega,\mathbf{x}^s) \;=\; & i\omega \int_\Sigma \frac{h_m(\mathbf{x}^r)\,a(\mathbf{x}^r,\mathbf{x})}{[\rho(\mathbf{x}^r)\,v(\mathbf{x}^r)]^{1/2}}\,\frac{c_{ijkl}(\mathbf{x})\,n_j}{\rho(\mathbf{x})[v^r(\mathbf{x})\,v^s(\mathbf{x})]^{1/2}} \\ \times\; & \Big\{h_i^r(\mathbf{x})h_k^{\text{spec}}(\mathbf{x})p_l^{\text{spec}}(\mathbf{x}) + h_k^r(\mathbf{x})h_i^{\text{spec}}(\mathbf{x})p_l(\mathbf{x})\Big\} \\ \times\; & e^{i\omega[T(\mathbf{x}^r,\mathbf{x})+T(\mathbf{x},\mathbf{x}^s)]}\,\frac{R(\mathbf{x},\mathbf{x}^s)\,a(\mathbf{x},\mathbf{x}^s)\,h_n(\mathbf{x}^s)}{[\rho(\mathbf{x}^s)\,v(\mathbf{x}^s)]^{1/2}}\,d\sigma\;. \end{aligned} \qquad (14)$$

This is the Kirchhoff-Helmholtz integral for anisotropic elastic media. The phase velocities at x with superscript s are taken in the direction of the downgoing ray from x^s to x, while the velocities with superscript r at x are taken in the direction of the upgoing ray form x to x^r.

In the next section, stationary-phase analysis for this integral shows that replacing the two phase velocities $v^r(\mathrm{x})$ and $v^s(\mathrm{x})$ in the scalar inner kernel above by the corresponding group velocities $V^r(\mathrm{x})$ and $V^s(\mathrm{x})$ yields the GRA expression of the reflected field from Σ due to point source at x^s and observed at x^r.

THE STATIONARY-PHASE APPROXIMATION

We want to compute the stationary values of the surface scattering integral of the type

$$I = i\omega \int_{\Sigma} b(\mathbf{x})\; e^{i\omega T(\mathbf{x})}\; d\boldsymbol{\sigma} \;, \tag{15}$$

assuming ω to be a large number ($|\omega| \gg 1$).

The stationary points satisfy

$$\frac{\partial T}{\partial \sigma_j} = \frac{\partial T}{\partial x_k}\;\frac{\partial x_k}{\partial \sigma_j} = \boldsymbol{\nabla} T \cdot \mathbf{t}_j \;= 0 \;, \quad i, j, = 1, 2 \;. \tag{16}$$

where $\mathbf{t}_j$, $j = 1, 2$ are the surface tangents. This condition is equivalent to Snell's law. Assume now that the stationary point is regular, so that $\det \mathbf{H} \neq 0$, where the matrix $\mathbf{H}$ has elements

$$H_{ij} = \frac{\partial^2 T}{\partial \sigma_i \partial \sigma_j} = \frac{\partial^2 T}{\partial x_n \partial x_k}\;\frac{\partial x_n}{\partial \sigma_i}\;\frac{\partial x_k}{\partial \sigma_j} + \frac{\partial T}{\partial x_k}\;\frac{\partial^2 x_k}{\partial \sigma_i \partial \sigma_j} \;, \quad i, j = 1, 2 \;. \tag{17}$$

Then the stationary value of the integral is (Bleistein, 1984, equation (2.8.23))

$$\tilde{I} = i\omega \left(\frac{2\pi}{|\omega|}\right)\; |\det \mathbf{H}|^{-1/2}\; e^{i\,\frac{\pi}{4}\,\mathrm{sgn}(\omega)\;\mathrm{Sgn}(\mathbf{H})}\; b(\tilde{\mathbf{x}})\; e^{i\omega T(\tilde{\mathbf{x}})} \;, \quad |\omega| \gg 1 \;, \tag{18}$$

where $\tilde{\mathbf{x}} = \mathbf{x}(\tilde{\boldsymbol{\sigma}})$ is the stationary point and $\mathrm{Sgn}(\mathbf{H})$ is the signature of the matrix $\mathbf{H}$, i.e., the difference between the number of its positive and the number of its negative eigenvalues.

The stationary point $\tilde{\mathbf{x}}$ is a point of specular reflection, so that $\mathbf{h}^{\mathrm{spec}}(\tilde{\mathbf{x}}) = \mathbf{h}^r(\tilde{\mathbf{x}})$ and $\mathbf{p}^{\mathrm{spec}}(\tilde{\mathbf{x}}) = \mathbf{p}^r(\tilde{\mathbf{x}})$. This gives rise to the following expression for the integral (14) after stationary-phase evaluation

$$\begin{aligned}\Delta G_{mn}(\mathbf{x}^r, \omega, \mathbf{x}^s) \;\simeq\;& 2\pi\; |\det \mathbf{H}|^{-1/2}\; e^{i\,\frac{\pi}{4}\,\mathrm{sgn}(\omega)\;[\mathrm{Sgn}(\mathbf{H})+2]}\; e^{i\omega[T(\tilde{\mathbf{x}},\mathbf{x}^s)+T(\tilde{\mathbf{x}},\mathbf{x}^r)]} \\ &\times\; \frac{h_m(\mathbf{x}^r)\; a(\mathbf{x}^r, \tilde{\mathbf{x}})}{[\rho(\mathbf{x}^r)\; v(\mathbf{x}^r)]^{1/2}}\; M(\tilde{\mathbf{x}})\; \frac{R(\mathbf{x}, \mathbf{x}^s)\; a(\tilde{\mathbf{x}}, \mathbf{x}^s)\; h_n(\mathbf{x}^s)}{[\rho(\mathbf{x}^s)\; v(\mathbf{x}^s)]^{1/2}}\; d\sigma \;.\end{aligned} \tag{19}$$

The nucleus M in equation (19) is given by

$$\begin{aligned}M(\tilde{\mathbf{x}}) \;&=\; \frac{2c_{ijkl}(\tilde{\mathbf{x}})\; h_i^r(\tilde{\mathbf{x}})\; h_n^r(\tilde{\mathbf{x}})\; p_l^r\; n_j}{\rho(\tilde{\mathbf{x}})[v^s(\tilde{\mathbf{x}})\; v^r(\tilde{\mathbf{x}})]^{1/2}} \\ &=\; \frac{2V_j^r(\tilde{\mathbf{x}})\; n_j}{[v^s(\tilde{\mathbf{x}})\; v^r(\tilde{\mathbf{x}})]^{1/2}} = -\;\frac{2V^r\;\cos\theta^r}{[v^s(\tilde{\mathbf{x}})\; v^r(\tilde{\mathbf{x}})]^{1/2}} \;,\end{aligned} \tag{20}$$

where we have used that (see., e.g., Cervený, 1995)

$$c_{ijkl}(\tilde{\mathbf{x}})\; h_i^r(\tilde{\mathbf{x}})\; h_n^r(\tilde{\mathbf{x}})\; p_l^r \;= V_j^r(\tilde{\mathbf{x}}) \tag{21}$$

is the jth component of the group velocity $\mathbf{V}^r(\tilde{\mathrm{x}})$ of the reflected ray in direction to the receiver. Also, $V^r = |\mathbf{V}^r|$ and θ^r is the angle between the normal and the ray from x^r to $\tilde{\mathrm{x}}$, as shown in Figure 1.

Interestingly enough, if we replace the phase velocities with the corresponding group velocities in the above expression, the result is

$$M = - \left[\frac{V^r}{V^s}\right]^{1/2} \cos\theta^r \ . \tag{22}$$

When this is used in the stationary-phase evaluation (19) of equation (14), we obtain the expression

$$\begin{aligned} \Delta\tilde{G}(\mathrm{x}^r, \omega; \mathrm{x}^s) &= \frac{\mathbf{h}(\mathrm{x}^r)}{[\rho(\mathrm{x}^r)v(\mathrm{x}^r)]^{1/2}} \left\{ \tilde{R}(\tilde{\mathrm{x}}, \mathrm{x}^s) \, \frac{e^{-i\,\frac{\pi}{2}\,\mathrm{sgn}(\omega)\,\sigma(\mathrm{x}^r,\tilde{\mathrm{x}},\mathrm{x}^s)}}{4\pi|\det \underset{\sim}{\boldsymbol{Q}}_2(\mathrm{x}^r, \tilde{\mathrm{x}}, \mathrm{x}^s)|^{1/2}} \right. \\ &\times \left. e^{i\omega T(\mathrm{x}^r,\tilde{\mathrm{x}},\mathrm{x}^s)} \right\} \frac{\mathbf{h}^T(\mathrm{x}^s)}{[\rho(\mathrm{x}^s)v(\mathrm{x}^s)]^{1/2}} \ . \end{aligned} \tag{23}$$

which has the form of the GRA Green's function for the reflected wave with the energy-normalized reflection coefficient

$$\tilde{R}(\tilde{\mathrm{x}}, \mathrm{x}^s) = R(\mathrm{x}, \mathrm{x}^s) \left[\frac{V^r(\tilde{\mathrm{x}}) \ \cos\theta^r}{V^s(\tilde{\mathrm{x}}) \ \cos\theta^s}\right]^{1/2} , \tag{24}$$

relative geometrical spreading factor

$$|\det \underset{\sim}{\boldsymbol{Q}}_2(\mathrm{x}^r, \tilde{\mathrm{x}}, \mathrm{x}^s)|^{1/2} = \left| \frac{\det \mathbf{H} \ \det \underset{\sim}{\boldsymbol{Q}}_2(\tilde{\mathrm{x}}, \mathrm{x}^r) \det \underset{\sim}{\boldsymbol{Q}}_2(\tilde{\mathrm{x}}, \mathrm{x}^s)}{\cos\theta^r \cos\theta^s} \right|^{1/2} , \tag{25}$$

KMAH index

$$\sigma(\mathrm{x}^r, \tilde{\mathrm{x}}, \mathrm{x}^s) = \sigma(\tilde{\mathrm{x}}, \mathrm{x}^r) + \sigma(\tilde{\mathrm{x}}, \mathrm{x}^s) + [1 - \mathrm{Sgn}(\mathbf{H})/2] \ , \tag{26}$$

and reflection traveltime

$$T(\mathrm{x}^r, \tilde{\mathrm{x}}, \mathrm{x}^s) = T(\tilde{\mathrm{x}}, \mathrm{x}^r) + T(\tilde{\mathrm{x}}, \mathrm{x}^s) \ . \tag{27}$$

These expressions were also obtained in Ursin and Tygel (1998).

CONCLUSIONS

We have used a generalized Kirchhoff-Helmholtz approximation in the Kirchhoff integral for anisotropic media. The upgoing field at the interface was replaced by the specularly reflected field, as approximated by the GRA. Within the validity of the GRA, the new integral formula can be used to compute multiply reflected and converted waves in anisotropic media. This also includes a possible wave-mode conversion at the interface.

The stationary-phase analysis of the new integral resulted in an approximative GRA Green's function. The amplitude at the receiver is given by a factor that includes the correct geometrical-spreading factor and a close approximation to the energy-normalized reflection coefficient. The exact GRA for the reflected wavefield is obtained by replacing the phase velocities with the group velocities in the source and receiver Green's functions at the scattering surface when computing the integral for the reflected wavefield.

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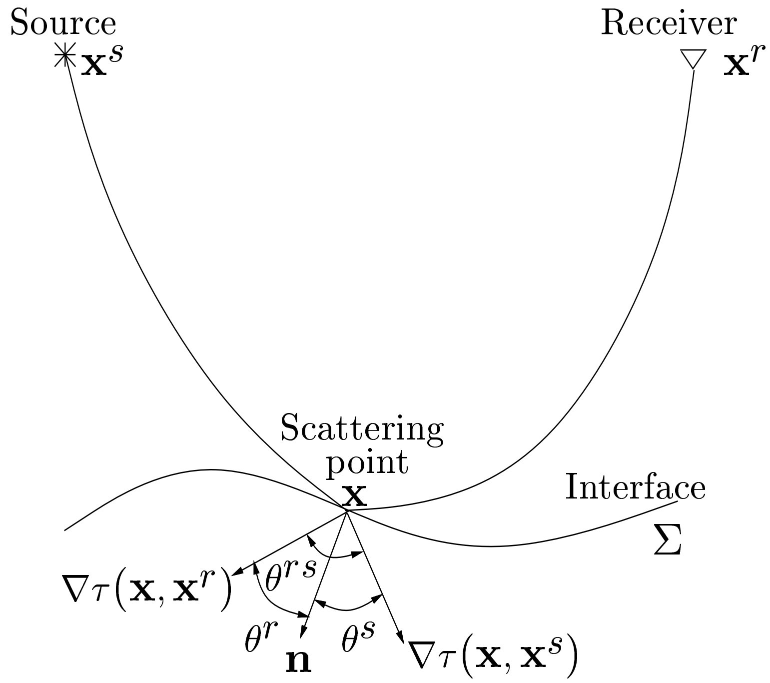

Figure 1: Geometry at the reflecting interface