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Calculation of effective velocities in cracked media using the rotated staggered finite-difference grid

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ABSTRACT

The modeling of elastic waves in fractured media with an explicit finite difference (FD) scheme causes instability problems on a staggered grid because the medium possesses high contrast discontinuities (strong heterogeneities). For the present study we apply the rotated staggered grid. Using this modified grid it is possible to simulate the propagation of elastic waves in a 2D or 3D medium containing cracks, pores or free surfaces without hard-coded boundary conditions. Therefore it allows an efficient and precise numerical study of effective velocities in fractured structures. We model the propagation of plane waves through a set of different randomly cracked media. In these numerical experiments we vary the wavelength of the plane waves, the porosity and the crack density. The synthetic results are compared with several theories that predict the effective P- and S-wave velocities in fractured materials. On the one hand for randomly distributed rectilinear non-intersecting thin dry cracks the numerical simulations of velocities of P-, SV- and SH-waves are in excellent agreement with the results of a modified (or differential) self-consistent theory. On the other hand for randomly distributed rectilinear intersecting thin dry cracks a classical differential theory, including a so called critical porosity, is the best way to describe the effective velocities.

INTRODUCTION

The problem of effective elastic properties of fractured solids is of considerable interest for geophysics, for material science, and for solid mechanics. Particularly, it is important for constitutive modeling of brittle micro cracking materials. For obvious reasons of practicality, the problem of three-dimensional medium permeated by circular or elliptical planar cracks has received more attention in literature. In this paper we consider the problem of a fractured medium in two dimensions. This may seem to be a significant oversimplification. However, we think that with this work some broad generalizations can be elucidated that will help solving problems with more complicated

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geometries.
Finite difference (FD) methods discretize the wave equation on a grid. They replace spatial derivatives by FD operators using neighboring points. The wave field is also discretized in time, and the wave field for the next time step is calculated in general by using a Taylor expansion. The main idea using a staggered grid is to calculate spatial derivatives halfway between two grid points to improve numerical accuracy. Hence, some modeling parameters are required to be defined on inter grid locations. Thus they have to be averaged or the grid values shifted halfway between two grid points have to be used. This yields for standard staggered grid schemes inaccurate results or instability problems. This is especially the case when the propagation of waves in media with strong fluctuations (e.g. cracks) of the elastic parameters is simulated, although the von Neumann stability (see e.g. Crase (1990)) is fulfilled. In the present numerical study, however, we apply the rotated staggered grid Saenger et al. (1999, 2000) for modeling of elastic wave propagation in arbitrary heterogeneous media. The rotated staggered grid is briefly discussed in the section ”Finite-difference modeling of fractured media”.
In this paper we present a numerical study of effective velocities of two types of fractured media. We model the propagation of a plane wave through a well defined fractured region. The numerical setup is described in section ”Experimental Setup”. We use randomly distributed rectilinear dry thin cracks in both media. For the first type of media we examine only non-intersecting cracks. The numerical results for P-, SV- and SH-waves (see section “Non-Intersecting Cracks”) are compared comprehensively with several theories Kachanov (1992); Davis and Knopoff (1995) that predict the effective velocities for such case. In the second type of fractured media we cancel the restriction of non-intersecting cracks. For this case the theories for non-intersecting cracks become out of their range of validity. However, a theory of Mukerji et al. (1995) , including a so called critical porosity, is in excellent agreement with our numerical results shown in section “Intersecting Cracks”.

FINITE-DIFFERENCE MODELING OF FRACTURED MEDIA

The propagation of elastic waves is described by the elastodynamic wave equation (e.g. Aki and Richards (1980)):

$$\rho(\mathbf{r})\ddot{u}_i(\mathbf{r}) = (c_{ijkl}(\mathbf{r})u_{k,l}(\mathbf{r}))_{,j} + f(\mathbf{r}). \quad (1)$$

For modeling elastic waves with finite-differences, it is necessary to discretize the stiffness tensor c_{ijkl}, the density ρ and the wave field u_i on a grid.

The standard staggered grid

A standard way of discretization of standard staggered grids (e.g. Kneib and Kerner (1993)) is shown in Figure 1a. The main reason for using this method is to improve numerical accuracy with respect to centered FD grids. Figure 1a shows the elementary cell, replacing, for example, the density at the left and the lower side by the density of the center and replacing the shear modulus at the lower left corner by the shear modulus at the center.
There is only one density location and one location for the Lamé parameter μ in an elementary cell. The calculation of the stress component σ_{xz} has to be done by multiplying the values of strain and stiffness defined at different positions. The same difficulties arises for the calculation of the acceleration, since the density has changed its location completely. When the wave field hits the inhomogeneities with high density contrasts (e.g. cracks), stability problems can also occur. Here we obtain an unstable modeling of a wave field diffraction on a crack Saenger et al. (2000). Note, that such stability problems exist although the von Neumann stability criterion is fulfilled.

The rotated staggered grid

All these difficulties described above can be avoided by choosing another configuration of the grid. Placing all components of the stiffness tensor at the same position within the elementary grid cell (e.g. the center), the positions of the modeling parameters are found directly as shown in Figure 1b. The directions of spatial derivatives have changed from x and z to $\tilde{\mathrm{x}}$ and $\tilde{\mathrm{z}}$.
The grid in Figure 1b satisfies all conditions with respect to the operations that are necessary to perform a time step. The parameters that have to be multiplied are defined at the same location and derivatives are defined between the parameters that have to be differentiated. Since the density is not located at the same position as the stiffness tensor elements, a density averaging (using the four surrounding cells) has also to be done for this grid. In the case of homogeneous cells or a linear behavior of the density between the stiffness locations, the density coincides with the exact density after averaging. The new distribution of elastic parameters is also advantageous for anisotropic modeling.

Stability and dispersion

Since FD modeling approximates derivatives by numerical operators and uses Taylor polynomials to perform the time update, inaccuracies occur, especially for coarse grids. One can separate these numerical errors into amplitude and phase errors. For a plane wave propagating through an infinite, isotropic and homogeneous medium, the amplitude must be conserved, and the velocity of propagation is not frequency-dependent. In

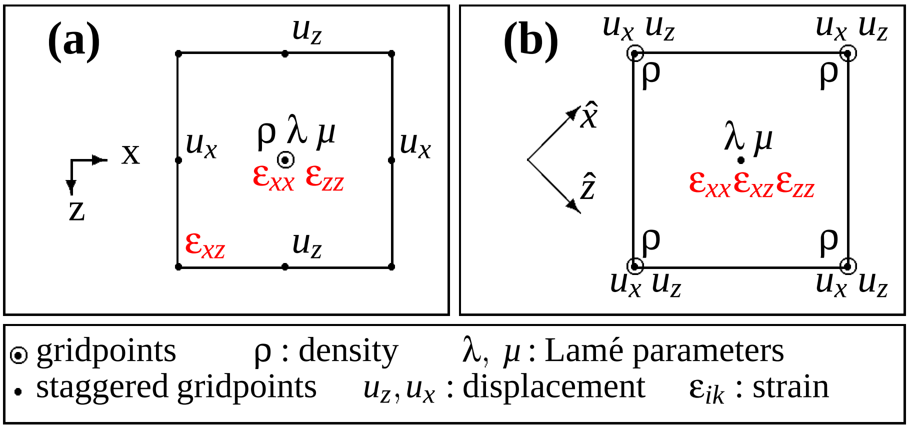

Figure 1: Elementary cells of different staggered grids. Locations where strains, displacement and elastic parameters are defined. **(a)** Locations on a standard staggered grid (e.g. Kneib and Kerner (1993)) if no averaging of medium parameters is performed. **(b)** Elementary cell of the rotated staggered grid. Spatial derivatives are performed along the $\tilde{x}$- and $\tilde{z}$-axes. The wave equation and the elements of the stiffness tensor are the same as in **(a)**.

FD modeling, it is possible that the amplitude increases exponentially with every time step. In this case, the modeling scheme is said to be unstable. Frequency-dependent velocity errors, also called numerical dispersion, cannot be excluded completely but can be estimated and, therefore, reduced to a known and acceptable degree. Note, that the von Neumann stability criterion is not in connection with the stability problems for high contrast inclusions. For the rotated grid the stability criterion for the 3D case and for the 2D case are the same. We obtain:

$$\frac{\Delta t v_p}{\Delta h} \leq 1/(\sum_{k=1}^{n} |c_k|). \tag{2}$$

In this equation c_k denotes the difference coefficients (e.g. Central Limit coefficients Karrenbach (1995), v_p the compressional wave velocity, Δt the time increment, and Δh the grid spacing. This result yields the von Neumann stability criterion for the rotated grid for all wave numbers in the case of homogeneous media and for 2nd order operator in time.

The dispersion error for the rotated staggered grid are similar to those of the conventional staggered grid. For a more detailed description of the rotated staggered grid refer to Saenger et al. (1999, 2000).

EXPERIMENTAL SETUP

As described above, the rotated staggered FD scheme is a powerful tool for testing theories about fractured media. There exist many theories that predict the effective elastic moduli of multiply fractured media as a function of crack density (e.g Davis and Knopoff (1995); Kachanov (1992)) or porosity (e.g. Mukerji et al. (1995); Norris (1985); Zimmermann (1991); Berryman (1992)). The elastic moduli and the velocities in a medium are related with well known formulas Aki and Richards (1980). In order to test such theories we create some elastic models with a region with a well known crack density (Eq. 3) or porosity. The cracked region was filled for this reason at random with randomly oriented cracks. In Figure 2 (left hand side) we can see a typical model with non-intersecting cracks. This model contains 1000 $\times$ 1910 grid points with an interval of 0.0001m. In the homogeneous region we set $v_p = 5100\frac{m}{s}$, $v_s = 2944\frac{m}{s}$ and $\rho_g = 2.7\frac{g}{cm^3}$. Table 1 is the summary of relevant parameters of all the models we use for our experiments. For the dry cracks we set $v_p = 0\frac{m}{s}$, $v_s = 0\frac{m}{s}$ and $\rho_g = 0.0000001\frac{g}{cm^3}$ which shall approximate vacuum Saenger et al. (1999, 2000). Therefore each additional crack increase the porosity.

It is very important to note that we perform our modeling experiments with periodic boundary conditions in the horizontal direction. For this reason our elastic models are generated also with this periodicity. Hence, it is possible that a single crack start at the right side of the model and ends at its left side.

To obtain the effective velocities in the different models we apply a body force line source at the top of the model. The plane wave in this way generated propagates through the fractured medium. With two lines of geophones at the top and at the bottom (see Figure 2) it is possible to measure the time-delay of the plane wave caused by the inhomogeneous region. With the time-delay one can calculate the effective velocity. Additionally, the attenuation of the plane wave can be fixed.

The direction of the body force and the source wavelet (source time function) can vary. Owing to this we can generate two types of shear (SH- and SV-) waves and one compressional (P-) wave. The source wavelet in our experiments is always the first derivative of a Gaussian with different dominant frequencies and with a time increment of $\Delta t = 5 * 10^{-9} s$. In Table 2 one can find details of the wavelets.

A very similar and successful experimental setup to test effective parameters in acoustic media can be found in Shapiro and Kneib (1993).

NON-INTERSECTING CRACKS

In this section we consider randomly distributed rectilinear non-intersecting thin dry cracks in 2D-media. We found papers Kachanov (1992) and Davis and Knopoff (1995) as good start to study such a case. Both papers discuss three different theories for 2D-media that predict an effective velocity for fractured models. Namely, they are the

No.	crack density ρ	length of cracks [0.0001m]	aspect ratio of cracks	number of cracks	porosity ϕ of the crack region	crack region [grid points]
1	0.199994	7	0.14	15280	0.1407	1000*1000
2	0.200013	14	0.077	3881	0.0708	1000*1000
3	0.200035	28	0.04	996	0.0358	1000*1000
4	0.024706	56	0.021	31	0.0022	1000*1000
5	0.050219	56	0.021	63	0.0046	1000*1000
6	0.100226	56	0.021	126	0.0091	1000*1000
7	0.200357	56	0.021	252	0.0181	1000*1000
8	0.200355	56	0.021	252	0.0181	500*2000
9	0.199987	112	0.011	126	0.0091	1000*1000
10x	0.050064	56	0.021	63	0.0045	1000*1000
11x	0.100054	56	0.021	126	0.0091	1000*1000
12x	0.200367	56	0.021	252	0.0181	1000*1000
13x	0.300409	56	0.021	378	0.0270	1000*1000
14x	0.400803	56	0.021	504	0.0360	1000*1000
15x	0.601186	56	0.021	756	0.0539	1000*1000
16x	0.800734	56	0.021	1007	0.0720	1000*1000

Table 1: This table contains information of the different crack models of the numerical study. The models with an x attached to its number have intersection of cracks.

No.	f_{fund} (Hz)	P- wavelength (fund.) [0.0001 m]	S- wavelength (fund.) [0.0001 m]
1	2200000	23	13
2	800000	64	37
3	400000	128	74
4	120000	425	245
5	50000	1020	588
6	22000	2318	1338

Table 2: Information of the different wavelets of the numerical study.

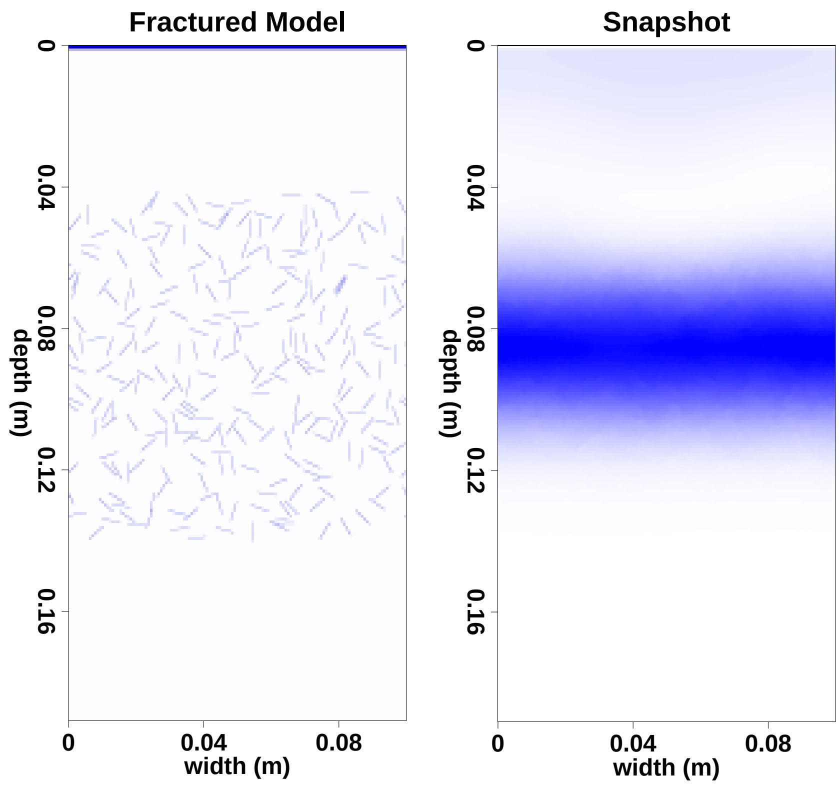

Figure 2: The left side shows a typical fractured model (No. 7) used for the numerical experiments. We introduce a cracked region in a homogeneous material. At the top we place a small strip of vacuum. This is advantageous for applying a body force line source with the rotated staggered grid. The right side is a snapshot of a plane wave propagating through the cracked region.

"theory for non-interacting cracks", the "self-consistent theory" and the "modified (or differential) self-consistent theory". Our goal is to compare the numerical results of the present study with the predicted effective velocities of the three theories.

In order to give an overview we summarize here the ideas and results for the three theories for one type of shear wave (SH- wave; vibration direction perpendicular to the 2D-model). For the definition of the crack density parameter we use (as Kachanov (1992)):

$$\rho = \frac{1}{A} \sum_{k=1}^{n} l_k^2 \tag{3}$$

(rectilinear cracks of length $2l_k$, A is the representative area).

The formula for the theory for non-interacting cracks is derived for the case of a dilute crack density. The energy per unit crack length, needed to insert a single antiplane crack, is added n times to the energy of the unfractured medium. With this assumption we can calculate the effective shear modulus $<\mu>$ Davis and Knopoff (1995):

$$<\mu>=\mu_0 \ \frac{1}{1+\pi*\rho/2}, \tag{4}$$

where μ_0 is the shear modulus of the unfractured medium and ρ is the crack density (Eq. 3).
In the simplest form of self-consistent calculations to determine the properties at higher orders, it is argued that an individual crack is introduced into an already cracked medium and hence should be subjected to the stress field in the flawed system and not to that in the unflawed system Budiansky and O'Connell (1976); O'Connell and Budiansky (1974, 1976). This yields the following prediction:

$$<\mu>=\mu_0 \ (1-\pi*\rho/2). \tag{5}$$

Two other papers argue that the change in energy should be calculated sequentially. This argument leads to the shear modulus as the solution to a simple differential equation Bruner (1976); Henyey and Pomphrey (1982), which is:

$$<\mu>=\mu_0 \ e^{-\pi*\rho/2}. \tag{6}$$

This is called the modified (or differential) self-consistent theory.
Note, that in first order all three theories predict the same effective modulus.

Numerical results

In this section we discuss the numerical results on effective wave velocities. They are depicted with dots in Figure 3. For comparison, the predictions of the three theories described above are shown in the same Figure with lines.
We show the normalized effective velocities for three types of waves. The relative decrease of the effective velocity for one given crack density is in the following succession: For SH-waves we obtain the smallest decrease followed by SV-waves. For P-waves it is largest. For each wave type we perform four numerical FD-calculations with different crack densities to obtain the effective velocity. For these measurements we only use the models No. 4,5,6,7 (see Tab. 1) and the wavelet No. 5 (see Tab. 2). Hence it follows the ratio of crack length to the dominant wavelength (Eq. 7): $p = 0.095$ for S-waves, and $p = 0.055$ for P-waves.
In Figure 3 the three dashed lines are due to the prediction by the theory for non-interacting cracks Kachanov (1992); Davis and Knopoff (1995). The three dashed-dotted lines are the prediction by the self-consistent theory Kachanov (1992); Davis and Knopoff (1995) and the three solid lines are to the prediction by the modified (or differential) self-consistent theory Kachanov (1992); Davis and Knopoff (1995). The top curves (red) are the results of a shear wave with vibration direction perpendicular

to the 2D-model (SH-waves). The other shear wave results (horizontal vibration direction, SV-waves) are in the middle (green). The bottom curves (blue) are the results for compressional (P-) waves.
If we follow the argumentation of Douma (1988), the aspect ratio of the cracks we used in our numerical experiments do not significantly influence the results of the three discussed theories. A final result is that our numerical simulations of P-, SV- and SH-wave velocities are in an excellent agreement with the predictions of the modified (or differential) self-consistent theory.
Now we want to examine the influence of the ratio of crack length to the dominant

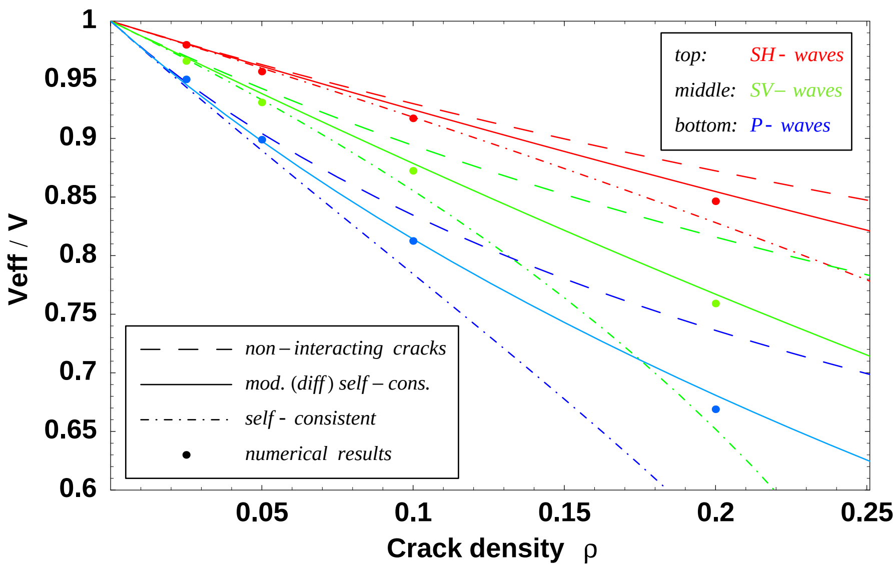

Figure 3: Normalized effective velocity versus crack density. The details are in section ”Numerical results”.

wavelength in our numerical experiments. The ratio is given by the parameter p:

$$p = \frac{2l}{\lambda_{\text{dom}}} \tag{7}$$

(rectilinear cracks of length $2l$, dominant wavelength λ_{dom}). We restrict ourself to study only the influence with one single crack density ($\rho = 0.2$) and for shear waves with vibration direction perpendicular to the 2D-model.
There are two possibilities to vary the parameter p. The first possibility is to vary λ_{dom} by using all wavelets in Table 2 and do not change the length of the cracks using models No. 7 and 8 (Table 1). It is important to note that the three theories mentioned above are only valid for wavelengths much larger than the crack length Peacock and Hudson (1990). With results shown in Figure 4 (dots joined with (blue) dashed line) we demonstrate that this is no restriction for our numerical calculations.
For the second curve in Figure 4 (dots joined with (red) dashed-dotted line) we always use wavelet No. 5 and vary the length of the cracks using models No. 1,2,3,7,8,9 (see

Table 2). Note, that with decreasing length of cracks the porosity is increasing. Therefore, the decrease of the effective velocity for small values of p in this curve can be explained by the increasing influence of the porosity of the used models.
The main result of both curves in Figure 4 is that our calculated effective velocities (dots in Figure 4) always match the prediction by the modified self consistent theory ((black) solid horizontal line) better than the prediction by the theory for non-interacting cracks ((black) dashed horizontal line) for all values of p. Thus, the values of p used for the numerical experiments depicted in Figure 3 are reasonable.

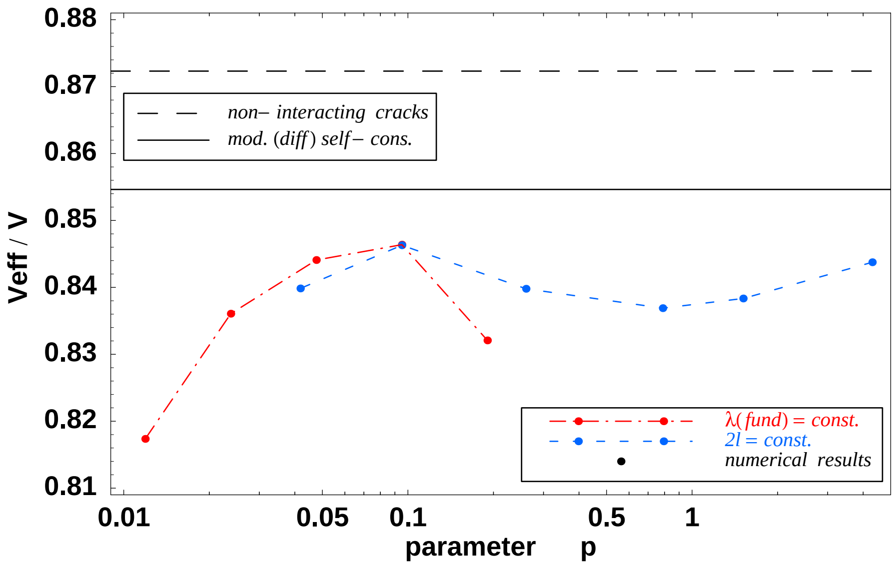

Figure 4: Normalized effective velocity versus $p = 2l / \lambda_{\text{dom}}$, for crack density $\rho = 0.2$. The details are in section ”Numerical results”.

INTERSECTING CRACKS

This section is about randomly distributed rectilinear intersecting thin dry cracks in 2D-media. The theories described in section “Non-Intersecting cracks” are not applicable in this case, because they are derived in particular for non-intersecting cracks. Therefore we have to use another theory for the comparison between prediction of the effective velocity and our numerical results. The differential effective medium (DEM) Norris (1985); Zimmermann (1991) formulations, including a critical porosity are appropriate in this case and can be found in Mukerji et al. (1995). This modified DEM model incorporating percolation behavior is always consistent with the Hashin-Shrikman bounds Hashin and Shtrikman (1963). Note, for this theory the effective velocities are predicted in dependence of porosity ϕ and not of crack density ρ.

The first step to use this theory is to determine the critical porosity. At this percolation porosity, the material is a loose packing of grains barely touching each other. With our models this value is easy to detect visually. For a porosity of $\phi_c = 0.13$ we found one horizontal circular way where the elastic parameters are set to the vacuum values.
The second step is to calculate the effective elastic parameters at the critical porosity. The moduli at the percolation point are equal to the Reuss (harmonic) average of the constituent moduli Mukerji et al. (1995). In our case we obtain:

$$\mu_c = \lim_{\mu_2 \to 0 \text{ Pa}} \frac{2}{1/\mu_1 + 1/\mu_2} = 0 \text{ Pa} \tag{8}$$

and

$$K_c = \lim_{K_2 \to 0 \text{ Pa}} \frac{2}{1/K_1 + 1/K_2} = 0 \text{ Pa}. \tag{9}$$

The third step is to calculate the effective bulk and shear moduli $K(y)$ and $\mu(y)$. We have to solve the following two coupled differential equations Berryman (1992):

$$(1-y)\frac{d}{dy}[K(y)] = [K_2 - K(y)]P(y) \tag{10}$$

$$(1-y)\frac{d}{dy}[\mu(y)] = [\mu_2 - \mu(y)]Q(y) \tag{11}$$

with initial conditions $K(0) = K_1$ and $\mu(0) = \mu_1$. For needle-like inclusions $P(y)$ and $Q(y)$ are Berryman (1980):

$$P(y) = \frac{\frac{1}{3}\mu_2 + K(y) + \mu(y)}{K_2 + \frac{1}{3}\mu_2 + \mu(y)}, \tag{12}$$

$$Q(y) = \frac{1}{5}\left(\frac{4\,\mu(y)}{\mu_2 + \mu(y)} + \frac{2\,[\mu(y) + \gamma(y)]}{\mu_2 + \gamma(y)} + \frac{K_2 + \frac{4}{3}\mu_2}{K_2 + \frac{1}{3}\mu_2 + \mu(y)}\right) \tag{13}$$

with:

$$\gamma(y) = \frac{\mu(y)\,[3\,K(y) + \mu(y)]}{3\,K(y) + 7\,\mu(y)}. \tag{14}$$

The main idea to involve the critical porosity in this equations is to set Mukerji et al. (1995): $K_2 = K_c$ and $\mu_2 = \mu_c$. With this definition, y denotes the concentration of the critical phase in the material and now the total porosity is $\phi = y\phi_c$. The effective velocities predicted by this theory are plotted in Figure 5.

Numerical results II

Our numerical results for intersecting cracks for shear waves with vibration direction perpendicular to the 2D-model (SH- waves) can be seen in Figure 5. For the calculations marked with dots we use always wavelet No. 5 (see Table 2). The models

No. 10x,11x,12x,13x,14x,15x,16x for experiments are in Table 1. In fact, our numerical results and the differential effective medium formulations, including a critical porosity, are in a very good agreement.
Note, that the theory described above is a 3D theory. With our 2D modeling results we can only test some special cases of this theory. For SH-waves one can say that our 2D crack model have needle-like inclusions.

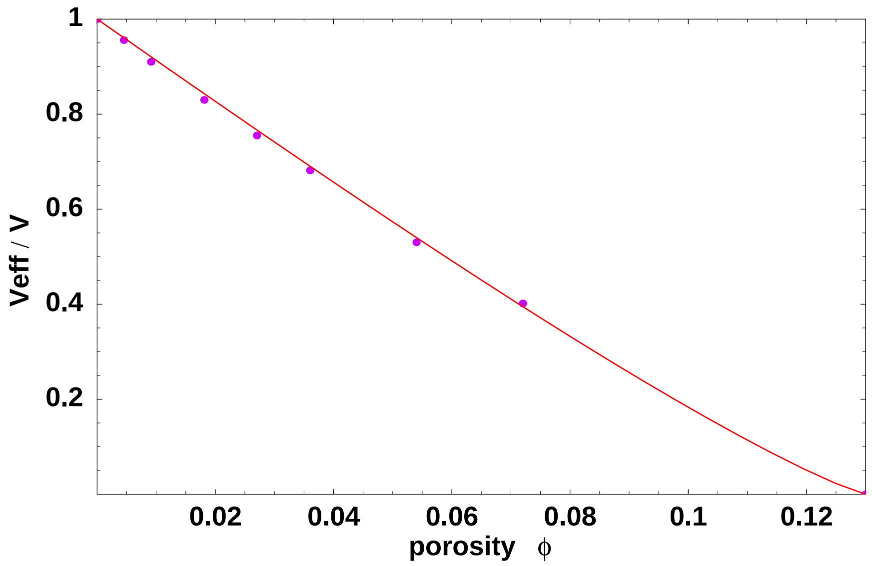

Figure 5: Normalized effective shear (SH-waves) velocity for intersecting cracks versus porosity. Dots: Numerical results, Solid Line: Theoretical prediction. The details are in section “Numerical results II”.

CONCLUSIONS

Since FD modeling discretize the medium and the wave field on a grid and not by finite volumes (like, e.g., Finite Element schemes), it requires very few assumptions. If the discretization is done correctly, FD modeling is very fast and accurate. In contrast to a standard staggered grid high-contrast inclusions do not cause instability difficulties for our rotated staggered grid. We demonstrated with our numerical calculations in this paper that our modification of the grid can model fractured media very well. For both cases, intersecting and non-intersecting cracks, the differential methods are very successful to predict effective velocities in fractured media.

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