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The Kirchhoff-Helmholtz integral for anisotropic elastic media

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ABSTRACT

The Kirchhoff-Helmholtz integral is a powerful tool to model the scattered wavefield from a smooth interface in acoustic or isotropic elastic media due to a given incident wavefield and observation points sufficiently far away from the interface. This integral makes use of the Kirchhoff approximation of the unknown scattered wavefield and its normal derivative at the interface in terms of the corresponding quantities that refer to the known incident field. An atractive property of the Kirchhoff-Helmholtz integral is that its asymptotic evaluation recovers the zero-order ray theory approximation of the reflected wavefield at all observation points where that theory is valid. Here, we extend the Kirchhoff-Helmholtz modeling integral to general anisotropic elastic media. It uses the natural extension of the Kirchhoff approximation of the scattered wavefield and its normal derivative for that media. The anisotropic Kirchhoff-Helmholtz integral also asymptotically provides the zero-order ray theory approximation of the reflected response from the interface. In connection with the asymptotic evaluation of the Kirchhoff-Helmholtz integral, we also derive an extension to anisotropic media of a useful decomposition formula of the geometrical spreading of a primary reflection ray.

INTRODUCTION

The field scattered from a smooth interface can be represented by an integral over the interface. Both the field and its normal derivative at the interface appear in the integrand. Fundamental representations for acoustic, isotropic elastic, and anisotropic elastic cases can be found in the literature (see, e.g., Aki and Richards, 1980; Bleistein, 1984). These representations can be recast as modeling formulas by exploiting the Kirchhoff approximation. This approximation expresses the (unknown) scattered field and its normal derivative in terms of corresponding (known) quantities of the incident field. For example, see Bleistein (1984) for the acoustic case and Frazer and Sen (1985)

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for the isotropic elastic case. The resulting modeling integral is called the Kirchhoff-Helmholtz integral. A fundamental feature of this integral is that the appication of the stationary-phase method yields zero-order ray theory.

In this paper, we derive the extension of the Kirchhoff-Helmholtz integral to anisotropic elastic media. The Kirchhoff approximation for this case follows along the same lines as the earlier derivations. For this case, the application of the stationary-phase method to compare to ray theory is not nearly as straightforward as in the previous cases. This is largely a consequence of the fact that the group and phase velocity vectors generally separate in anisotropic media.

The stationary-phase formula requires the evaluation of the determinant of the matrix of second derivatives of the traveltime function. The challenge here is to express this determinant in terms of geometrical spreading factors that characterize the ray-theoretic solution. This leads to a decomposition of the geometric-spreading of a primary-reflection ray as a product of three factors: one is the point-source geometrical-spreading factor of the incidence ray; another is the point-source geometrical-spreading factor of the reflected ray; the last is the so-called Fresnel geometrical-spreading factor that accounts for the influence of the interface to the overall spreading. Such a decomposition already exists for the acoustic and the isotropic elastic cases (Hubral et al., 1992; Tygel et al., 1994; Cervený, 1995). Here it is extended to anisotropic elastic media. The result confirms that the Kirchhoff-Helmholtz integral has zero-order ray theory as its leading order approximation.

We first derive the anisotropic Kirchhoff-Helmholtz integral from basic principles. We then compute its asymptotic evaluation using the stationary-phase method and state the geometrical-spreading decomposition formula.

THE KIRCHHOFF-HELMHOLTZ INTEGRAL

In the frequency domain, wave propagation in an inhomogeneous anisotropic elastic solid, in the absence of sources, is governed by the elastic Helmholtz equation (Aki and Richards, 1980)

$$-\,\omega^2 \rho u_i - (c_{ijkl} u_{k,l})_{,j} = 0 \; , \tag{1}$$

where $u_i = u_i(\boldsymbol{x}, \omega)$ is the i-th component of the displacement vector $\boldsymbol{u}(\boldsymbol{x})$, $\rho = \rho(\boldsymbol{x})$ is the density and $c_{ijkl} = c_{ijkl}(\boldsymbol{x})$ are the elastic parameters of the medium at the point $\boldsymbol{x} = (x_1, x_2, x_3)$. Also, ω is the temporal frequency. The elastic parameters satisfy the symmetry relations $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. In equation (1) the notation "$, j$" stands for $\partial/\partial x_j$. Also, a repeated index implies summation with respect to this index.

The Green's function, $g_{in}(\boldsymbol{x}, \omega; \boldsymbol{x}^s)$, satisfies the equation

$$-\,\omega^2 \rho g_{in} - (c_{ijkl} g_{kn,l})_{,j} = \delta_{in} \delta(\boldsymbol{x} - \boldsymbol{x}^s) \; . \tag{2}$$

It also satisfies the reciprocity relation (Aki and Richards, 1980, equation (2.39))

$$g_{ij}(\boldsymbol{x},\omega;\boldsymbol{x}^s) = g_{ji}(\boldsymbol{x}^s,\omega;\boldsymbol{x}) \ . \tag{3}$$

We shall use the geometric ray approximation (GRA) to obtain the approximate Green's function. Using the results in Cervený (1995), we have for a specific ray connecting a source point $\boldsymbol{x}^s$ to a scattering point $\boldsymbol{x}$, the GRA Green's function

$$g_{ij}(\boldsymbol{x},\omega;\boldsymbol{x}^s) = h_i^s(\boldsymbol{x}) \ \frac{a(\boldsymbol{x},\boldsymbol{x}^s)\ e^{i\omega T(\boldsymbol{x},\boldsymbol{x}^s)}}{[\rho(\boldsymbol{x})v^s(\boldsymbol{x})\rho(\boldsymbol{x}^s)v(\boldsymbol{x}^s)]^{1/2}} \ h_j(\boldsymbol{x}^s) \ , \tag{4}$$

where $\boldsymbol{h}(\boldsymbol{x}^s)$ and $\boldsymbol{h}^s(\boldsymbol{x})$ are the unit polarization vectors, $\rho(\boldsymbol{x}^s)$ and $\rho(\boldsymbol{x})$ are the densities and $v(\boldsymbol{x}^s)$ and $v^s(\boldsymbol{x})$ the phase velocities in the ray direction at the source $\boldsymbol{x}^s$ and at the point $\boldsymbol{x}$, respectively. Moreover, $T(\boldsymbol{x},\boldsymbol{x}^s)$ is the traveltime along the ray from $\boldsymbol{x}$ to $\boldsymbol{x}^s$ and

$$a(\boldsymbol{x},\boldsymbol{x}^s) = \frac{e^{-i\frac{\pi}{2}\,\mathrm{sgn}(\omega)\,\kappa(\boldsymbol{x},\boldsymbol{x}^s)}}{4\pi|\det \boldsymbol{Q}_2(\boldsymbol{x},\boldsymbol{x}^s)|^{1/2}} \tag{5}$$

is a complex amplitude function taking into account possible caustics and phase-shift at the source. In this expression, $|\det \boldsymbol{Q}_2(\boldsymbol{x},\boldsymbol{x}^s)|^{1/2}$ denotes the relative geometric spreading factor and $\kappa(\boldsymbol{x},\boldsymbol{x}^s)$ is the KMAH index for the ray that connects the source $\boldsymbol{x}^s$ to the point $\boldsymbol{x}$.

We remark that our notation for the phase velocities $v^s(\boldsymbol{x})$ and $v(\boldsymbol{x}^s)$ require some explanation. These are both velocities along the specific ray that connects $\boldsymbol{x}^s$ to $\boldsymbol{x}$. Due to anisotropy, these velocities depend on the ray direction, as well as position. When we check reciprocity and interchange $\boldsymbol{x}$ and $\boldsymbol{x}^s$, we must also shift the superscript s. In an implicit way, this reflects the fact that these velocities are indepent of the direction of traversal of the ray. Our notation for the polarization vectors $h_i^s(\boldsymbol{x})$ and $h_i(\boldsymbol{x}^s)$ follows the same pattern. As a consequence, the reciprocity relation (3) also holds for the GRA Green's function (4).

We shall approximate the spatial derivatives by their leading-order terms in powers of ω, namely

$$\begin{aligned} g_{ij,k}(\boldsymbol{x},\omega;\boldsymbol{x}^s) &\approx i\omega\, T_{,k}(\boldsymbol{x},\boldsymbol{x}^s)\, g_{ij}(\boldsymbol{x},\omega;\boldsymbol{x}^s) \\ &= i\omega\, p_k^s g_{ij}(\boldsymbol{x},\omega;\boldsymbol{x}^s) \end{aligned} \tag{6}$$

where $p_k^s = p_k^s(\boldsymbol{x}) = T_{,k}(\boldsymbol{x},\boldsymbol{x}^s)$ is the kth component of the slowness vector $\boldsymbol{p}^s(\boldsymbol{x})$ at the point $\boldsymbol{x}$ (for the ray from the source).

We shall consider an incident wavefield, $u_i^{inc}(\boldsymbol{x},\omega)$, that is being reflected from a surface Σ and recorded at the point $\boldsymbol{x}^r$, as shown in Figure 1. The reflected wavefield at $\boldsymbol{x}^r$, $u_i^{\text{refl}}(\boldsymbol{x}^r,\omega)$, can be expressed as a surface integral involving the displacement field $u_i^{\text{refl}}(\boldsymbol{x},\omega)$ and its partial derivatives $u_{k,l}^{\text{refl}}(\boldsymbol{x},\omega)$ at the surface Σ by using a representation theorem that is given in Aki and Richards (1980), equation (2.41). In the absence of body forces, and with a Green's function that satisfies the reciprocity relation (3), this representation is

$$\begin{aligned} u_m^{\text{refl}}(\boldsymbol{x}^r,\omega) &= \int_\Sigma \Big\{ g_{mi}(\boldsymbol{x}^r,\omega,\boldsymbol{x})\, c_{ijkl}(\boldsymbol{x})\, u_{k,l}^{\text{refl}}(\boldsymbol{x},\omega) \\ &- g_{mk,l}(\boldsymbol{x}^r,\omega,\boldsymbol{x})\, c_{ijkl}(\boldsymbol{x})\, u_i^{\text{refl}}(\boldsymbol{x},\omega) \Big\} n_j \, d\sigma \, . \end{aligned} \tag{7}$$

To transform the above representation integral into a modeling formula, we need to replace the unknown wavefields $u_i^{\text{refl}}(\boldsymbol{x},\omega)$ and $u_{i,j}^{\text{refl}}(\boldsymbol{x},\omega)$ by suitable known approximations of those wavefields. We assume that the incident wavefield, at each point of the interface Σ, is given in the GRA form

$$u_i^{inc}(\boldsymbol{x},\omega) = h_i^{inc}(\boldsymbol{x})\, A^{inc}(\boldsymbol{x})\, \exp[i\omega\, T^{inc}(\boldsymbol{x})] \, , \tag{8}$$

Here, $h_i^{inc}(\boldsymbol{x})$, $A^{inc}(\boldsymbol{x})$ and $T^{inc}(\boldsymbol{x})$ are the polarization vector, the amplitude and the traveltime, respectively, of a specific wave mode selection of the incidence field. Of course, the complete GRA description of that wavefield requires the consideration of all relevant wave modes for the modeling problem under consideration. All those wave modes have the same form of equation (8). By the linearity of the integral representation, the total response is obtained by simple superposition. It suffices, thus, to consider just one incident wave mode of the form (8).

Generalizing Bleistein's (1984) formulas for acoustic media and those of Frazer and Sen (1985) for the elastic media, we propose to replace $u_i^{\text{refl}}(\boldsymbol{x},\omega)$ in the anisotropic Kirchhoff approximation by

$$u_i^{\text{refl}}(\boldsymbol{x},\omega) \approx h_i^{\text{spec}}(\boldsymbol{x})\, R(\boldsymbol{x},\boldsymbol{p}^{inc})\, A^{inc}(\boldsymbol{x})\, \exp[i\omega\, T^{inc}(\boldsymbol{x})] \, , \tag{9}$$

Here, $h_i^{\text{spec}}(\boldsymbol{x})$ is the polarization vector corresponding to a specular reflected wave of proper mode at the point $\boldsymbol{x}$ on Σ, due to the incident wave and $R(\boldsymbol{x},\boldsymbol{p}^{inc})$ is the plane-wave reflection coefficient (normalized with respect to displacement amplitude) for our choice of incoming and outgoing type of wave. All other factors are as defined in connection with equation (4). As above, we considered just one wave mode of the reflected field and rely on linearity to allow us to construct the total reflected field as a simple sum.

Note that a corresponding approximation has been mentioned by de Hoop and Bleistein (1997). In this paper, we complete the details of their observation and demonstrate that it yields the leading-order anisotropic ray solution for the reflected wave after the application of the stationary-phase method.

The anisotropic Kirchhoff approximation of the spatial derivatives of the reflected wavefield at point $\boldsymbol{x}$ will also follow the pattern of their acoustic and isotropic elastic counterparts. Namely, we set

$$\begin{aligned} u_{k,l}^{\text{refl}}(\boldsymbol{x},\omega) &\approx i\omega\, p_l^{\text{spec}}(\boldsymbol{x})\, u_k^{\text{refl}}(\boldsymbol{x},\omega) \\ &= i\omega\, p_l^{\text{spec}}(\boldsymbol{x})\, h_k^{\text{spec}}(\boldsymbol{x})\, R(\boldsymbol{x},\boldsymbol{p}^{inc})\, A^{inc}(\boldsymbol{x})\, \exp[i\omega\, T^{inc}(\boldsymbol{x})] \, , \end{aligned} \tag{10}$$

where $\boldsymbol{p}^{\text{spec}}(\boldsymbol{x})$ is the slowness vector of the specular reflected wave, that is, it is related to $\boldsymbol{p}^s$ by Snell's law for plane waves incident on planar reflectors.

To construct the Kirchhoff-Helmholtz modeling formula based on the representation integral (7), we shall need the GRA Green's function $g_{mi}(\boldsymbol{x}^r, \omega, \boldsymbol{x})$ from the point $\boldsymbol{x}$ to the receiver at $\boldsymbol{x}^r$ for another specified wave (also given by a specified ray code). We shall also need the partial derivatives $g_{mk,j}(\boldsymbol{x}^r, \omega, \boldsymbol{x})$ of that GRA Green's function. The expressions of these two functions can be readily derived upon an obvious choice of arguments in equations (4) and (6), respectively. A possible wave-mode conversion at $\boldsymbol{x}$ is taken care of by selecting the proper reflection coefficient in equation (9). With the Kirchhoff approximation as represented by equations (9) and 10), equation (7) can be approximated by

$$
\begin{aligned}
u_m^{KH}(\boldsymbol{x}^r, \omega) \;&=\; i\omega \int_\Sigma \frac{h_m(\boldsymbol{x}^r)}{[\rho(\boldsymbol{x}^r)\, v(\boldsymbol{x}^r)]^{1/2}} \frac{c_{ijkl}(\boldsymbol{x})\, n_j}{[\rho(\boldsymbol{x})\, v^r(\boldsymbol{x})]^{1/2}} \\
&\times \left\{ h_i^r(\boldsymbol{x})\, p_l^{\text{spec}}(\boldsymbol{x})\, h_k^{\text{spec}}(\boldsymbol{x}) - p_l^r(\boldsymbol{x})\, h_k^r(\boldsymbol{x})\, h_i^{\text{spec}}(\boldsymbol{x}) \right\} \\
&\times e^{i\omega[T(\boldsymbol{x}^r, \boldsymbol{x}) + T^{inc}(\boldsymbol{x})]}\, a(\boldsymbol{x}^r, \boldsymbol{x})\, A^{inc}(\boldsymbol{x})\, R(\boldsymbol{x}, \boldsymbol{p}^{inc})\, d\sigma
\end{aligned} \tag{11}
$$

This is the Kirchhoff-Helmholtz integral that models the reflected wavefield from an interface in an inhomogeneous anisotropic elastic media due to an arbitrary incident wavefield.

The above-obtained Kirchhoff-Helmholtz integral is of particular interest when the incident wavefield is chosen to be the GRA Green's function of equation (4). In that case, that modeling integral becomes a useful approximation of the reflected response of the interface due to a point-source excitation, namely an approximation of the reflected Green's function from that interface. The explicit expression of the Kirchhoff-Helmholtz integral in this important situation is readily found to be

$$
\begin{aligned}
g_{mn}^{KH}(\boldsymbol{x}^r, \omega) \;&=\; i\omega \int_\Sigma \frac{h_m(\boldsymbol{x}^r)}{[\rho(\boldsymbol{x}^r)\, v(\boldsymbol{x}^r)]^{1/2}} \frac{c_{ijkl}(\boldsymbol{x})\, n_j}{\rho(\boldsymbol{x})\, [v^r(\boldsymbol{x})\, v^s(\boldsymbol{x})]^{1/2}} \\
&\times \left\{ h_i^r(\boldsymbol{x})\, p_l^{\text{spec}}(\boldsymbol{x})\, h_k^{\text{spec}}(\boldsymbol{x}) - p_l^r(\boldsymbol{x})\, h_k^r(\boldsymbol{x})\, h_i^{\text{spec}}(\boldsymbol{x}) \right\} \\
&\times e^{i\omega\, T(\boldsymbol{x}^r, \boldsymbol{x}, \boldsymbol{x}^s)}\, a(\boldsymbol{x}^r, \boldsymbol{x}, \boldsymbol{x}^s)\, R(\boldsymbol{x}, \boldsymbol{p}^s)\, \frac{h_n(\boldsymbol{x}^s)}{[\rho(\boldsymbol{x}^s)\, v(\boldsymbol{x}^s)]^{1/2}}\, d\sigma \ ,
\end{aligned} \tag{12}
$$

where we have used the simplifying notation

$$
T(\boldsymbol{x}^r, \boldsymbol{x}, \boldsymbol{x}^s) = T(\boldsymbol{x}^r, \boldsymbol{x}) + T(\boldsymbol{x}, \boldsymbol{x}^s) \quad \text{and} \quad a(\boldsymbol{x}^r, \boldsymbol{x}, \boldsymbol{x}^s) = a(\boldsymbol{x}^r, \boldsymbol{x})\, a(\boldsymbol{x}, \boldsymbol{x}^s)\, . \tag{13}
$$

In the next section, the stationary-phase analysis of this integral shows that the high-frequency asymptotic evaluation yields the GRA expression of the reflected field from Σ due to point source at $\boldsymbol{x}^s$ and observed at $\boldsymbol{x}^r$.

THE STATIONARY-PHASE APPROXIMATION

We want to compute the stationary values of the surface scattering integral of the type

$$I = i\omega \int_{\Sigma} b(\boldsymbol{x})\, e^{i\omega T(\boldsymbol{x})}\, d\boldsymbol{\sigma} \ , \tag{14}$$

to leading order in the high-frequency ω.

The stationary points $\tilde{\boldsymbol{x}}$ satisfy

$$\frac{\partial T}{\partial \sigma_j} = \frac{\partial T}{\partial x_k}\,\frac{\partial x_k}{\partial \sigma_j} = \boldsymbol{\nabla} T \cdot \boldsymbol{t}_j \ = 0 \ , \quad i, j, = 1, 2 \ . \tag{15}$$

where $\boldsymbol{t}_j$, $j = 1, 2$ are the surface tangents. This condition is equivalent to Snell's law. For simplicity, we assume that there is only one stationary point $\tilde{\boldsymbol{x}}$. Furthermore, we assume that the stationary point is regular, so that $\det \boldsymbol{H} \neq 0$, where the matrix $\boldsymbol{H}$ has elements

$$H_{ij} = \frac{\partial^2 T}{\partial \sigma_i \partial \sigma_j} = \frac{\partial^2 T}{\partial x_n \partial x_k}\,\frac{\partial x_n}{\partial \sigma_i}\,\frac{\partial x_k}{\partial \sigma_j} \ , \qquad i, j = 1, 2 \ , \tag{16}$$

evaluated at $\tilde{\boldsymbol{x}}$. Then the stationary value of the integral is (Bleistein, 1984, , equation (2.8.23))

$$\tilde{I} = i\omega \left(\frac{2\pi}{|\omega|}\right) \ |\det \boldsymbol{H}|^{-1/2} \ e^{i\,\frac{\pi}{4}\,\mathrm{sgn}(\omega)\,\mathrm{Sgn}(\boldsymbol{H})}\, b(\tilde{\boldsymbol{x}})\, e^{i\omega T(\tilde{\boldsymbol{x}})} \ , \tag{17}$$

where $\tilde{\boldsymbol{x}} = \boldsymbol{x}(\tilde{\boldsymbol{\sigma}})$ is the stationary point and $\mathrm{Sgn}(\boldsymbol{H})$ is the signature of the matrix $\boldsymbol{H}$, that is, the difference between the number of its positive eigenvalues and the number of its negative eigenvalues.

The stationary point $\tilde{\boldsymbol{x}}$ is a point of specular reflection, so that $\boldsymbol{h}^{\mathrm{spec}}(\tilde{\boldsymbol{x}}) = \boldsymbol{h}^r(\tilde{\boldsymbol{x}})$ and $\boldsymbol{p}^{\mathrm{spec}}(\tilde{\boldsymbol{x}}) = \boldsymbol{p}^r(\tilde{\boldsymbol{x}})$. This gives rise to the following expression for the integral (12) after stationary-phase evaluation

$$\begin{aligned} g_{mn}^{KH}(\boldsymbol{x}^r, \omega, \boldsymbol{x}^s) \ &\simeq \ 2\pi \ |\det \boldsymbol{H}|^{-1/2} \ e^{i\,\frac{\pi}{4}\,\mathrm{sgn}(\omega)\,[\mathrm{Sgn}(\boldsymbol{H})+2]} \\ &\times \ \frac{h_m(\boldsymbol{x}^r)}{[\rho(\boldsymbol{x}^r)\, v(\boldsymbol{x}^r)]^{1/2}}\ \frac{2c_{ijkl}(\tilde{\boldsymbol{x}})\, h_i^r(\tilde{\boldsymbol{x}})\, h_k^r(\tilde{\boldsymbol{x}})\, p_l^r\, n_j}{\rho(\tilde{\boldsymbol{x}})[v^s(\tilde{\boldsymbol{x}})\, v^r(\tilde{\boldsymbol{x}})]^{1/2}} \\ &\times \ R(\boldsymbol{x}, \boldsymbol{x}^s)\, a(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s)\, e^{i\omega\, T(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s)}\ \frac{h_n(\boldsymbol{x}^s)}{[\rho(\boldsymbol{x}^s)\, v(\boldsymbol{x}^s)]^{1/2}} \ . \end{aligned} \tag{18}$$

Let us now introduce the group velocities $V^{s,r} = |\boldsymbol{V}^{s,r}(\tilde{\boldsymbol{x}})|$ that pertain to the ray segments connecting the source $\boldsymbol{x}^s$ and the receiver $\boldsymbol{x}^r$, respectively, to the reflection point $\tilde{\boldsymbol{x}}$. We recall that the group velocity is a vector in the ray direction. We denote by $\alpha^{s,r}$ and $\theta^{s,r}$, respectively, the angles that the group and phase velocities $\boldsymbol{V}^{s,r}$ and $\boldsymbol{v}^{s,r}$ make with the normal $\boldsymbol{n}$ of the reflector at $\tilde{\boldsymbol{x}}$. In anisotropic media, the ray and slowness directions differ in general, that is, $\alpha^{s,r} \neq \theta^{s,r}$. Moreover, the three vectors $\boldsymbol{V}^{s,r}$, $\boldsymbol{v}^{s,r}$ and $\boldsymbol{n}$ do not, in general, lie in the same plane (see Figure 2).

Introducing the angle $\chi^{s,r}$ between the phase and group velocities $\boldsymbol{V}^{s,r}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{v}^{s,r}(\tilde{\boldsymbol{x}})$, we can express the relationship between these quantities as (see de Hoop and Bleistein, 1997)

$$v^{s,r}(\tilde{\boldsymbol{x}}) = V^{s,r}(\tilde{\boldsymbol{x}})\, \cos\chi^{s,r} \ . \tag{19}$$

Moreover, we also have that (see., e.g., Cervený, 1995)

$$\frac{c_{ijkl}(\tilde{\boldsymbol{x}})}{\rho(\tilde{\boldsymbol{x}})}\ h_i^r(\tilde{\boldsymbol{x}})\, h_n^r(\tilde{\boldsymbol{x}})\, p_l^r\, n_j = V_j^r(\tilde{\boldsymbol{x}})\, n_j = V^r(\tilde{\boldsymbol{x}}) \cos\alpha^r \ , \tag{20}$$

where $V^r = |\boldsymbol{V}^r|$ is the group velocity of the ray from the reflection point to the receiver. The geometry of the reflection point is shown in Figure 1.

When expressions (19) and (20) are used in the stationary result (19) of the Kirchhoff-Helmholtz integral, this reduces to

$$\boldsymbol{g}^{KH}(\boldsymbol{x}^r, \omega, \boldsymbol{x}^s) \simeq \boldsymbol{g}^{R}(\boldsymbol{x}^r, \omega, \boldsymbol{x}^s) \ , \tag{21}$$

where

$$\begin{aligned} \boldsymbol{g}^{R}(\boldsymbol{x}^r, \omega, \boldsymbol{x}^s) \ &= \ \frac{\boldsymbol{h}(\boldsymbol{x}^r)}{[\rho(\boldsymbol{x}^r)\, v(\boldsymbol{x}^r)]^{1/2}} \left\{ R^{\text{En}}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)\ \frac{e^{-i\,\frac{\pi}{2}\,\text{sgn}(\omega)\,\kappa(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s)}}{4\pi |\det \boldsymbol{Q}_2(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s)|^{1/2}} \right. \\ &\times \ \left. e^{i\omega T(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s)} \right\} \frac{\boldsymbol{h}^T(\boldsymbol{x}^s)}{[\rho(\boldsymbol{x}^s)\, v(\boldsymbol{x}^s)]^{1/2}} \end{aligned} \tag{22}$$

is the GRA Green's function for the reflected wave from $\boldsymbol{x}^s$ to $\boldsymbol{x}^r$. It includes the energy-normalized reflection coefficient

$$R^{\text{En}}(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) = R(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)\ \left[\frac{V^r(\tilde{\boldsymbol{x}})\ \cos\alpha^r}{V^s(\tilde{\boldsymbol{x}})\ \cos\alpha^s} \right]^{1/2} \ , \tag{23}$$

relative geometrical spreading factor

$$|\det \boldsymbol{Q}_2(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s)|^{1/2} = \left| \det \boldsymbol{H}\ \det \boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^r) \det \boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \frac{\cos\chi^r \cos\chi^s}{\cos\alpha^r \cos\alpha^s} \right|^{1/2} , \tag{24}$$

KMAH index,

$$\kappa(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s) = \kappa(\tilde{\boldsymbol{x}}, \boldsymbol{x}^r) + \kappa(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) + [1 - \text{Sgn}(\boldsymbol{H})/2] \ , \tag{25}$$

and reflection traveltime

$$T(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s) = T(\tilde{\boldsymbol{x}}, \boldsymbol{x}^r) + T(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s) \ . \tag{26}$$

Note that for each preassigned wave mode that arrive at $\boldsymbol{x}^r$, there is a construction of the form (22). The total wavefield at $\boldsymbol{x}^r$ is, f course, the sum of all of the relevant wave modes that make up the reflected response.

In acoustic and elastic isotropic media, the phase and group velocities coincide, implying that $\alpha^{s,r} = \theta^{s,r}$ and $\chi^{s,r} = 0$. As a consequence, the expression (24) for the relative geometrical spreading reduces to

$$|\det \boldsymbol{Q}_2(\boldsymbol{x}^r, \tilde{\boldsymbol{x}}, \boldsymbol{x}^s)|^{1/2} = \left| \frac{\det \boldsymbol{H} \ \det \boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^r) \det \boldsymbol{Q}_2(\tilde{\boldsymbol{x}}, \boldsymbol{x}^s)}{\cos\theta^r \cos\theta^s} \right|^{1/2} . \qquad (27)$$

This simplified expression was proposed by Ursin and Tygel (1997). It has to be corrected for its more general counterpart (24). Note that one can derive a general formula for the decomposition of the matrix $\boldsymbol{Q}_2(\boldsymbol{x}, \boldsymbol{x}^s)$ from ray-theoretical considerations. This result proves that equation (24) agrees with the reflected wavefield as derived by ray theory.

CONCLUSIONS

In this paper, we have extended the Kirchhoff-Helmholtz integral that is well-known for acoustic and elastic isotropic media, to generally anisotropic media. As was done in the cases of acoustic and isotropic elastic media, the upgoing, scattered field at the interface was replaced by the specularly reflected field, as approximated by the GRA. Within the validity of the GRA, the new integral formula can be used to compute multiply reflected and converted waves in anisotropic media. This also includes a possible wave-mode conversion at the interface. The present approach provides a "single-event" approximation that enables us to determine one specifically chosen reflection without having to calculate all other events that might be considered noise in the actual problem. This is, of course, no restriction, since the complete wavefield at the receiver is just the superposition of all possible events that can be calculated independently (but simultaneously, if so desired) by the corresponding Kirchhoff-Helmholtz integrals.

We have also extended the decomposition formula for the relative geometrical spreading factor from acoustic and elastic isotropic to anisotropic media. This generalization has been done independently, based only on ray-theoretical arguments. The resulting decomposition formula provides the means to calculate the geometrical spreading of a primary reflected ray in terms of the spreading factors of the incident and reflected ray segments and a third factor that accounts for the influence of the interface.

The generalized geometrical-spreading decomposition was crucial to show that the stationary-phase analysis of the new integral results in the GRA Green's function of the reflected wavefield. This comparison confirms that the generalized Kirchhoff approximation for the reflected field and its normal derivative at the reflector is correct in generally anisotropic elastic media.

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PUBLICATIONS

A paper containing these results has been submitted to Wave Motion.