Divergence of geometrical optics series at the boundary of its applicability: two analytical examples in elementary functions

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ABSTRACT

Two analytical examples in elementary functions are presented, which demonstrate divergence of the geometrical optics (GO) series when the conditions for its applicability are violated: (i) shear wave propagation in 1D elastic media with exponentially changing parameters and (ii) 2D Gaussian beam diffraction in free space. These examples evidence that accounting for higher terms in GO power series leads to divergence and therefore becomes completely senseless beyond the boundaries of GO applicability.

INTRODUCTION

Geometrical optics (or geometrical acoustics in the specific case of sound and/or elastic waves) is the most efficient and universal method in the wave theory. It deals with a representation of the wave in the form of series in inverse powers of the wave number $k_0 = \omega/v_0$:

$$u_{GO} = Ae^{ik_0\psi} = \left(A_0 + \frac{A_1}{ik_0} + \frac{A_2}{(ik_0)^2} + ...\right) e^{ik_0\psi} \tag{1}$$

Here, A denotes amplitude, ψ an eikonal, ω a frequency and v_0 is a typical wave velocity for the considered medium.

It is well known that convergence of the geometrical optics (GO) series (1) to the exact solution has an asymptotic nature. Asymptotic convergence means that the difference between the exact solution u_{exact} and the GO solution (1) tends to zero for $kL \to \infty$, where L is a characteristic scale of the medium and the wave field parameters, $k = nk_0$ and $n = v_0/v$ is a refractive index.

However, in contrast to the limit $kL \to \infty$, for sufficiently low frequencies the series (1) can diverge. A careful analysis of the asymptotic convergence requires significant efforts incorporating special functions or complicated integrals. This paper illustrates the convergence behavior of GO series by two examples using only elementary (i.e. algebraic and exponential) functions. The first example deals with a 1D shear wave propagating in a medium with exponentially increasing medium parameters (shear modulus and density). The second example describes the diffraction of a Gaussian beam in a free space. In both cases the wave equation has exact solutions in elementary functions which can be compared with the GO solution (1).

The new aspect of our analysis is the observation that the GO series starts diverging just when the conditions for its applicability are violated. The *necessary* condition for the validness of (1) is that the wave length λ is small compared to the characteristic scale L of the medium (Landau and Lifshits, 1977; Born and Wolf, 1983; Kravtsov and Orlov, 1990):

$$\frac{\lambda}{2\pi L} = \frac{1}{kL} = \frac{1}{nk_0 L} << 1\,, \tag{2}$$

where $n = v_0/v$ stands for the refractive index of the medium.

The *sufficient* condition for the validness of (1) is that the variations of the amplitude A and the refractive index n are small within a Fresnel volume (Kravtsov and Orlov, 1990, 1994; Kravtsov, 1988)

$$a_f|\nabla A| << A \qquad , \qquad a_f|\nabla n| << n \,. \tag{3}$$

Here, a_f is a cross section of the Fresnel volume which incorporates all first Fresnel zones surrounding the infinitely thin 'mathematical' ray. The Fresnel volume itself forms a 'physical' ray of finite thickness.

The first example (1D shear wave, Sect. 2) demonstrates the divergence of the GO series at

$$kL \approx 1 \,, \tag{4}$$

when the necessary condition (2) becomes violated. The second example (2D Gaussian beam diffraction, Sect. 3) shows that the condition (3) fails when the distance z exceeds the diffraction length $z_{diff} = k_0 a^2$, where a is an initial Gaussian beam width.

Both examples evidence that accounting for higher order terms in the series (1) beyond the boundary of GO validity is the reason for divergences instead of expected improvements of accuracy.

SHEAR WAVES IN 1D ELASTIC MEDIA

Geometrical optics expansion

Here, we treat the general case of a shear wave propagating in an elastic medium. The propagation of a time harmonic ($e^{-i\omega t}$) shear wave in a 1D medium with density $\rho(z)$ and shear modulus $\mu(z)$ is described by the wave equation

$$\frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) = \rho \frac{\partial^2 u}{\partial t^2} \,. \tag{5}$$

u denotes the displacement in the direction perpendicular to the z axis (so called SH case (Cerveny, 2001)).

A time harmonic high-frequency ansatz for the displacement u

$$u = Ae^{ik_0\psi} \tag{6}$$

leads to the equation

$$k_0^2 \mu A \left[\left(\frac{\partial \psi}{\partial z}\right)^2 - n^2\right] \tag{7}$$

$$-ik_0 \left[A\frac{\partial \mu}{\partial z}\frac{\partial \psi}{\partial z} + \mu A \frac{\partial^2 \psi}{\partial z^2} + 2\mu \frac{\partial A}{\partial z}\frac{\partial \psi}{\partial z}\right] \tag{8}$$

$$-\left[\frac{\partial \mu}{\partial z}\frac{\partial A}{\partial z} + \mu \frac{\partial^2 A}{\partial z^2}\right] = 0 \,, \tag{9}$$

where $n = v_0/v$ is the refractive index and $v = \sqrt{\mu/\rho}$ is the shear wave velocity. A similar formalism can be applied also to longitudinal (acoustic) waves using only slightly changed notations for the material parameters (for details see Buske (2000)).

The three terms in equations (7)-(9) are sorted according to the powers of k_0. The largest term (7), which is proportional to k_0^2, yields the eikonal equation

$$\left(\frac{\partial \psi}{\partial z}\right)^2 = n^2. \tag{10}$$

The subsequent terms (8) and (9) lead to transport equations for the amplitudes A_m of the GO expansion (1):

$$2\mu \frac{\partial A_0}{\partial z}\frac{\partial \psi}{\partial z} + \mu A_0 \frac{\partial^2 \psi}{\partial z^2} + A_0 \frac{\partial \mu}{\partial z}\frac{\partial \psi}{\partial z} = 0 \qquad\qquad m = 0 \,, \tag{11}$$

$$2\mu \frac{\partial A_m}{\partial z} \frac{\partial \psi}{\partial z} + \mu A_m \frac{\partial^2 \psi}{\partial z^2} + A_m \frac{\partial \mu}{\partial z} \frac{\partial \psi}{\partial z} = -\left(\frac{\partial \mu}{\partial z} \frac{\partial A_{m-1}}{\partial z} + \mu \frac{\partial^2 A_{m-1}}{\partial z^2} \right) \qquad m = 1, 2, ... \tag{12}$$

In a special case when density ρ and shear modulus μ grow exponentially with depth z

$$\rho = \rho_0 \exp(\gamma z) \qquad \mu = \mu_0 \exp(\gamma z) \tag{13}$$

the refractive index n becomes unity and the eikonal $\psi = z$. Under these conditions the transport equation (11) for the amplitude of the zeroth approximation has the solution

$$A_0(z) = A_0^0 \exp\left(-\frac{\gamma z}{2}\right), \tag{14}$$

where $A_0^0 = A_0(z = 0)$ is an initial value of A_0 at z=0. Thus, in the zeroth approximation of GO the solution of the wave equation becomes

$$u_{GO} = A_0^0 \exp\left(ik_0 z - \frac{\gamma z}{2}\right). \tag{15}$$

The modified transport equation

Simply combining the second term (8) and the third term (9) into a single equation one obtains the so called *modified transport equation* (MTE):

$$\frac{\partial^2 \tilde{A}}{\partial z^2} + \frac{\partial \mu}{\partial z} \frac{\partial \tilde{A}}{\partial z} + ik \left[\tilde{A} \frac{\partial \mu}{\partial z} \frac{\partial \psi}{\partial z} + \mu \tilde{A} \frac{\partial^2 \psi}{\partial z^2} + 2\mu \frac{\partial \tilde{A}}{\partial z} \frac{\partial \psi}{\partial z} \right] = 0. \tag{16}$$

From the solution $\tilde{A}$ of the MTE and with ψ from the eikonal equation (10) one can obtain the solution of the wave equation (5) in the form of (6). In the case of the exponentially changing parameters (13) the MTE takes the form

$$\frac{\partial^2 \tilde{A}}{\partial z^2} + [\gamma + 2ik_0] \frac{\partial \tilde{A}}{\partial z} + i\gamma k_0 \tilde{A} = 0, \tag{17}$$

and has the exact solution

$$\tilde{A} = a_1 \exp(q_1 z) + a_2 \exp(q_2 z). \tag{18}$$

The two solutions q_1 and q_2 of the corresponding characteristic equation are

$$q_1 = -\frac{\gamma}{2} - ik_0 + \sqrt{(ik_0)^2 + \left(\frac{\gamma}{2}\right)^2} \qquad q_2 = -\frac{\gamma}{2} - ik_0 - \sqrt{(ik_0)^2 + \left(\frac{\gamma}{2}\right)^2}. \tag{19}$$

The amplitude coefficients in equation (18) are

$$a_1 = \frac{\tilde{B}^0 - \tilde{A}^0 q_2}{q_1 - q_2} \quad , \quad a_2 = \frac{-\tilde{B}^0 + \tilde{A}^0 q_1}{q_1 - q_2} \tag{20}$$

where

$$\tilde{A}^0 = \tilde{A}(z = 0) \quad , \quad \tilde{B}^0 = \frac{\partial \tilde{A}(z = 0)}{\partial z} \tag{21}$$

are the initial values (at $z = 0$) of the amplitude $\tilde{A}(z)$ and its derivative $\tilde{B}(z) = d\tilde{A}(z)/dz$, respectively. For a given problem the necessary condition (2) for GO applicability can be written as inequality

$$k >> \gamma, \tag{22}$$

which means that the characteristic length $L \approx 1/\gamma$ of the variation in medium parameter is much larger than the wavelength:

$$L >> \frac{1}{k} = \frac{1}{nk_0}\,. \tag{23}$$

Under the condition $k >> \gamma$, when GO is applicable, the two solutions become

$$q_1 \approx -\frac{\gamma}{2} \qquad q_2 \approx -2ik_0\,. \tag{24}$$

In this case

$$q_1 - q_2 = 2\sqrt{(ik_0)^2 + \left(\frac{\gamma}{2}\right)^2} \approx 2ik_0\,, \tag{25}$$

$$a_1 \approx \frac{-\tilde{B}^0 + \tilde{A}^0 2ik_0}{2ik_0} \qquad a_2 \approx \frac{-\tilde{B}^0 - \tilde{A}^0 \frac{\gamma}{2}}{2ik_0}\,, \tag{26}$$

so that the solution of (17) takes (at $\tilde{B}^0 = 0$) the form:

$$\tilde{A} \approx \tilde{A}^0 \exp\left(-\frac{\gamma z}{2}\right) - \tilde{A}^0 \frac{\gamma}{4ik_0} \exp\left(-2ik_0 z\right). \tag{27}$$

The first term in (27) corresponds to the zeroth approximation of the GO solution (15) with natural interchange $A_0^0 \leftrightarrow \tilde{A}^0$, whereas the second term in (27) describes the reflected wave

$$u_{refl} \approx \tilde{A}_{refl} \exp\left(-ik_0 z\right) \tag{28}$$

with the amplitude $\tilde{A}_{refl}$ being proportional to the small factor $\gamma/k_0 << 1$.

Thus, the GO solution (15) in fact is the leading term of the exact solution (18), expanded into series in inverse powers of the wave number k_0. According to the theory of complex variables (Cartan, 1995) the power series for $q_1 - q_2$ takes the form

$$q_1 - q_2 = 2ik_0\sqrt{1 + \left(\frac{\gamma}{2ik_0}\right)^2} = 2ik_0\left[1 + \frac{1}{2}\left(\frac{\gamma}{2ik_0}\right)^2 - \frac{1}{8}\left(\frac{\gamma}{2ik_0}\right)^4 + ...\right] \tag{29}$$

and converges if only

$$\frac{\gamma}{2} < k_0\,, \tag{30}$$

that is exactly within the area of GO applicability. At the same time the GO series (1) *diverges* beyond the area of GO applicability. This means, from a practical point of view, that accounting for higher order terms in the GO series (1) makes sense only until inequality (30) holds, otherwise higher order terms will be the reason for the divergence of the GO series.

It is worth to notice that the reflected wave (the second term in equation (27)) principally cannot be expanded into a series in inverse powers of k_0. Therefore, the reflected wave can in no way be extracted from the GO expansion (1).

2D GAUSSIAN BEAM DIFFRACTION IN FREE SPACE

Exact solution of the wave equation in paraxial approximation

Let us consider 2D Gaussian beam diffraction in free space on the basis of paraxial approximation. If the initial wave field in the $z = 0$ plane is given by

$$u^0(x) = A^0 \exp\left(-\frac{x^2}{2a^2}\right) \quad , \quad a >> \lambda\,. \tag{31}$$

Then, the wave field at an arbitrary point (x, z) can be determined from the parabolic equation

$$2ik_0\frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial x^2} = 0 \quad , \quad u = U(x,z)e^{ik_0 z} \tag{32}$$

or from the Kirchhoff integral written in paraxial approximation

$$u(x,z) = \sqrt{\frac{i}{\lambda z}} \int u^0(\xi) \exp\left(ik_0 z + \frac{ik_0 (x-\xi)^2}{2z} \right) d\xi\,. \tag{33}$$

For an initial Gaussian wave field (31) one obtains from equations (32) and (33)

$$u_{exact}(x,z) = \frac{A^0}{\sqrt{1-iQ}} \exp\left(ik_0 z - \frac{x^2}{2a^2(1-iQ)} \right), \tag{34}$$

where

$$Q = \frac{z}{k_0 a^2} \tag{35}$$

denotes the ratio of the distance z to the diffraction length $z_{diff} = k_0 a^2$.

Geometrical optics solution for a 2D Gaussian beam

The geometrical optics solution for a 2D Gaussian beam has the form

$$u_{GO} = \left(A_0 + \frac{A_1}{ik_0} + \frac{A_2}{(ik_0)^2} + ... \right) e^{ik_0 z}\,. \tag{36}$$

Here, the amplitude of the zeroth approximation A_0 describes *undiffracted* Gaussian beam propagation

$$A_0(x,z) = A^0 \exp\left(-\frac{x^2}{2a^2} \right). \tag{37}$$

The higher order amplitude terms are responsible for diffraction and are given by the recursive formula

$$A_{n+1} = -\frac{1}{2} \int \frac{\partial^2 A_n}{\partial x^2}\, dz\,. \tag{38}$$

For a central ray $x = 0$ each differentiation with respect to x gives an additional factor $-1/a^2$ so that the amplitude terms read

$$\frac{A_1}{ik_0} = -\frac{zA^0}{2ik_0 a^2} = -\frac{iQ}{2}\,, \tag{39}$$

$$\frac{A_2}{(ik_0)^2} = \frac{3}{8}(iQ)^2\,, \tag{40}$$

$$\frac{A_3}{(ik_0)^3} = -\frac{5}{16}(iQ)^3\,. \tag{41}$$

Relation between GO solution and exact solution

One can compare the paraxial solution (34) with the GO series (36) by expanding the amplitude factor

$$(1-iQ)^{-\frac{1}{2}} \tag{42}$$

in equation (34) into power series in Q:

$$\left(1 + \frac{iQ}{2} + \frac{3}{8}(iQ)^2 - \frac{5}{16}(iQ)^3 + ... \right)\,. \tag{43}$$

One can see that the first three terms in (43) and in the GO series (39)-(41) are identical. However, it is a bit more troublesome to show the total identity of both series. According to the theory of functions of complex arguments (e.g. Cartan (1995)) the series (43) converges only if $Q < 1$. That means for $Q > 1$

the sum (43) will become infinite although the left hand side of (43) is bounded. The divergence of the GO series for $Q > 1$ can be illustrated by considering the squared normalized amplitude

$$S(Q) = \left[\frac{A(0,z)}{A^0}\right]^2 = \frac{1}{1 - iQ} \tag{44}$$

with its power series

$$S(Q) = \left(1 + iQ + (iQ)^2 + (iQ)^3 + ...\right) . \tag{45}$$

The sum of the first n terms of this series can be written in a more compact form

$$S_n(Q) = \frac{(iQ)^{n+1} - 1}{iQ - 1} . \tag{46}$$

For $Q < 1$ this series converges for $n \to \infty$ to the initial value (44), but for $Q > 1$ it diverges.

The difference between the exact solution (34) and the GO series (36) can be characterized by the normalized ratio

$$\delta_1(Q) = \frac{|u_{exact}(0,z) - u_{GO}(0,z)|}{A^0} , \tag{47}$$

which is shown in Figure 1. This difference is zero for $Q < 1$ but it turns out to be infinite for $Q > 1$.

A completely different plot is obtained for the difference between the exact solution $u_{exact}(0,z)$ and the zeroth order GO approximation

$$u^0_{GO}(0,z) = A_0(0,z) \exp\left(ik_0 z\right) . \tag{48}$$

The difference, presented in the normalized form

$$\delta_2(Q) = \frac{\left|u_{exact}(0,z) - u^0_{GO}(0,z)\right|}{A^0} \tag{49}$$

is shown in Figure (2). The value $\delta_2(Q)$ is close to zero at small distances, where $Q << 1$. At the boundary of GO applicability ($Q = 1$) the error takes the value

$$\left|\frac{1}{\sqrt{1 - iQ}} - 1\right| \tag{50}$$

and finally it tends to unity for $Q \to \infty$.

The divergence of the series (45) starts at

$$Q = \frac{z}{k_0 a^2} = 1 , \tag{51}$$

which is the convergence radius for the function $1/\sqrt{1 - iQ}$. The equality (51) corresponds to the sufficient criterion of GO inapplicability (see inequalities (3)): at $Q = 1$ the radius of the first Fresnel zone $a_f = \sqrt{\lambda z}$ becomes comparable with the width a of the primary Gaussian beam (31).

It is not surprising that the GO expansion (36) fails to describe diffraction phenomena for $Q > 1$. However, it looks surprising that taking into account higher terms in the GO expansion (36), which actually make the difference between the GO approximation and the exact solution and which are negligible for $Q < 1$, lead to unlimited growth of the GO field for $Q > 1$. The comparison of Fig. 1 and Fig. 2 prompts to conclude that accounting for higher order terms in the GO series (36) makes sense only if $Q < 1$, whereas for $Q > 1$ higher terms can only worsen the difference between the exact solution u_{exact} and the zeroth order GO approximation u^0_{GO}.

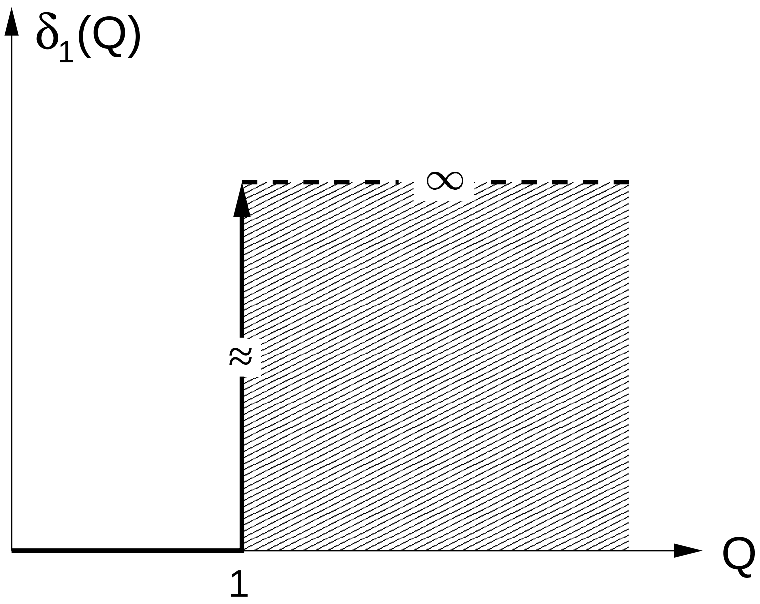

Figure 1: Normalized difference $\delta_1(Q)$, eq. (47), between the exact solution $u_{exact}(0, z)$ and the total GO solution $u_{GO}(0, z)$.

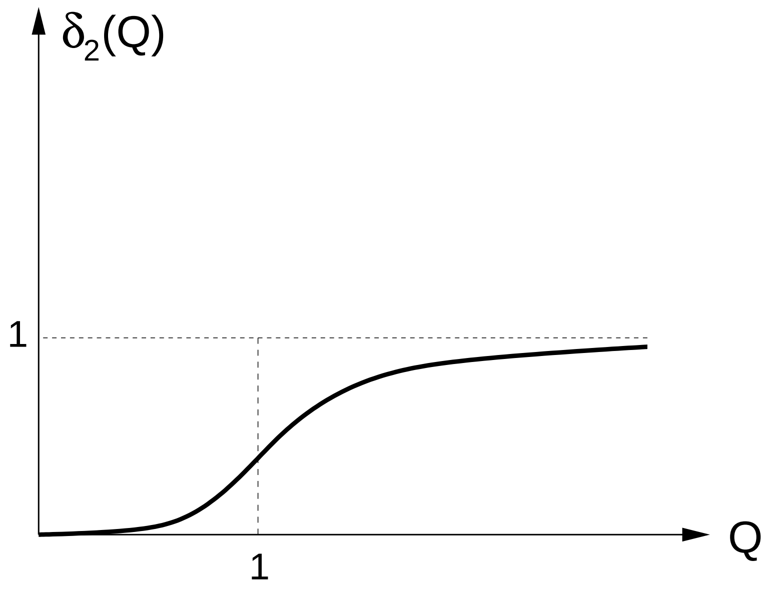

Figure 2: Normalized difference $\delta_2(Q)$, eq. (49), between the exact solution $u_{exact}(0, z)$ and the zeroth order GO approximation $u^0_{GO}(0, z)$.

CONCLUSIONS

Two analytical examples have shown explicitly that crossing the boundary of GO applicability might result not only in worsening the accuracy but also in catastrophical divergence of the GO series. Such a divergence reflects the asymptotic nature of the GO series in powers of the inverse wave number k_0. From a practical point of view this phenomenon restricts the possibilities to improve the accuracy of wave field calculations by accounting for higher order terms of the GO series beyond the area of GO applicability.

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