Numerical Rock Physics: Effects of parallel crack distributions on effective elastic properties

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ABSTRACT

This paper is concerned with numerical studies of effective elastic properties of cracked solids. We concentrate on two dimensional media containing different patterns of parallel crack distributions. We use the Rotated Staggered Grid (RSG) which allows one to simulate elastic wave propagation very accurately in fractured media. Our aim is to compare the predictions given by several effective medium theories to the effective properties we derive from our numerical experiments. Namely, these are the "Non-interaction approximation (NIA)", the "Differential scheme (DS)" and an extension of the DS (EDS). According to our results, the DS theory and its extension perform well. Simulations of media containing very few cracks prove that for our setup the effective properties stabilise at low numbers of cracks. Finally, we studied parallel cracks clustered in stacked columns. We found that, as expected, the shielding effects dominate in such patterns.

INTRODUCTION

Understanding the relations between crack microstructure and effective elastic properties of a solid has always been of great interest. Theoretical predictions of these relations often differ significantly. Exact knowledge of real crack microstructure is usually not available, making a direct comparison between theoretical predictions and experiment data difficult. Numerical simulations allow one to determine the effective properties of solids with known microstructures. They are therefore attractive for testing effective medium theories, if the simulated microstructures correspond to the assumptions of the theories. Here we rely on a finite difference (FD) scheme, the *Rotated Staggered Grid*, which simulates the propagation of elastic waves very accurately, even for models containing high contrasts in elastic parameters. A detailed description of the procedure is given by Saenger et al. (2000).

We present a numerical study of effective wave velocities and of the corresponding elastic moduli of two dimensional fractured media. First, we consider parallel cracks with uncorrelated positions, oriented perpendicular to the direction of shear and compressive wave propagation. The obtained results are compared to the ones of several effective media theories. According to our results, the DS theory performs well. Second, two families of parallel cracks, at an angle of $30°$ to each other, are placed at random in the model area. Waves propagate along the symmetry axes in the two dimensional plane. We find the agreement between our data and the EDS theory to be partly acceptable. Finally, we study models with randomly located clusters of five parallel cracks stacked in columns. Spacings between cracks within the clusters are varied in order to examine the effects of crack interactions in these particular patterns on effective velocities.

THEORY

This section gives an overview of the effective media theories relevant for our simulations. We consider two dimensional models with randomly distributed parallel cracks. As we work within the plane strain framework, our two dimensional plane can be understood as one of the symmetry planes of a three dimensional orthotropic solid. Seismic velocities are linked to the elastic moduli by (see, for example, Mavko et al. (1998), from where notations are taken):

$$c_{11} = \rho_g v_P^2(0°), \qquad c_{22} = \rho_g v_P^2(90°), \qquad c_{44} = \rho_g v_{SV}^2(90°) = \rho_g v_{SV}^2(0°) \tag{1}$$

where ρ_g denotes the gravitational density, c_{pq} are elements of the stiffness matrix and $v_P,\ v_{SV}$ are the phase velocities.

The stiffness matrix is given by the inverse of the compliance matrix:

$$\begin{bmatrix} <c_{11}> & <c_{12}> & 0 \\ <c_{12}> & <c_{22}> & 0 \\ 0 & 0 & <c_{44}> \end{bmatrix} = \begin{bmatrix} \frac{1-\nu^2}{<E_1>} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{1-\nu^2}{<E_2>} & 0 \\ 0 & 0 & \frac{1}{<G>} \end{bmatrix}^{-1}. \tag{2}$$

$E,\ \nu$ and G denote the Young's modulus, the Poisson's ratio and the shear modulus of the solid, respectively. Brackets $<>$ indicate effective moduli as given by effective media theories described in the following sections.

A common parameter to characterise cracked solids is the *crack density* ρ, which is given by (Bristow, 1960):

$$\rho = \frac{1}{A}\sum_{i=1}^{N} l_i^2. \tag{3}$$

A denotes the reference area, l_i the half length of the i-th crack and N is the number of cracks in A.

One family of parallel cracks

For models with one family of parallel cracks (see Fig. 1), Kachanov (1993) discusses three different theoretical descriptions which can be applied here - the NIA, the DS and the EDS theory. We overview briefly the underlying concepts of these theories.

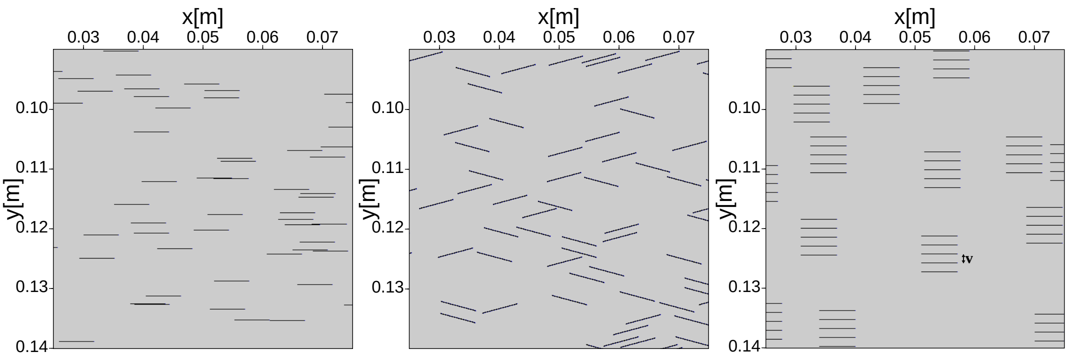

Figure 1: Part of cracked regions for $\rho = 0.2$ with randomly distributed parallel cracks (model 4p, see Tab. 1), left, and of the cracked region for $\rho = 0.2$ containing two families of parallel cracks, at an angle of $30°$ (model 2f, see Tab. 1), middle. On the right, a part of the cracked region with columnar clusters of five cracks (model 2.2b, see Tab. 2).

Non-interaction approximation (NIA) This approximation assumes that to obtain the elastic potential of a solid with cracks, the energy which is needed to insert a single crack into the unfractured media can simply be added to the elastic potential for each crack [as in Bristow (1960)]. The effective moduli are (cracks are parallel to the x-axis):

$$E_{NIA,1} = E_0',\ E_{NIA,2} = E_0'[1 + 2\pi\rho]^{-1},\ G_{NIA} = G_0[1 + \pi\rho(1-\nu_0)]^{-1} \tag{4}$$

where E_0' denotes $E_0/(1-\nu_0^2)$, the Young's modulus for the plane strain case.

Differential scheme (DS) Here, analysis is done incrementally: crack density is increased in small steps $d\rho$, effective matrix moduli are recalculated after each step. The effective moduli are:

$$E_{DS,1} = E_0', \quad E_{DS,2} = E_0' e^{-2\pi\rho}, \quad G_{DS} = G_0 e^{-\pi(1-\nu_0)\rho}. \tag{5}$$

Extension of the DS (EDS) This model (Kachanov, 1993) extends the DS to arbitrary crack orientation statistics by multiplying the elastic potential given by NIA by the function $g(\rho)$ which is based on results of the DS theory. This yields the following effective moduli for cracks parallel to the x-axis:

$$E_{EDS,1} = E_0',\ E_{EDS,2} = \frac{E_0'}{1 + 2\pi\rho e^{\pi\rho}},\ G_{EDS} = \frac{G_0}{1 + \pi\rho(1-\nu_0)e^{\pi\rho}}. \tag{6}$$

Two families of parallel cracks

We consider two families of parallel cracks of equal density inclined at $30°$ to each other (see Fig. 1). For this arrangement, as the orientation of cracks needs to enter the theoretical prediction explicitly, we apply only the NIA and the EDS theory.

If the coordinate system is oriented in such a way that its axes coincide with the principal axes of the crack orientation distribution as in Fig. 1, one obtains, for the effective moduli according to the NIA:

$$\begin{aligned} E_{NIA,1} = E_0'[1 + \pi\rho(1 + \sqrt{3}/2)]^{-1}, \quad E_{NIA,2} &= E_0'[(1 + \pi\rho(1 - \sqrt{3}/2))]^{-1}, \\ G_{NIA} &= G_0[1 + \pi\rho(1-\nu_0)]^{-1}. \end{aligned} \tag{7}$$

The EDS theory yields

$$\begin{aligned} E_{EDS,1} = \frac{E_0'}{1 + \pi\rho e^{\pi\rho}(1 + \sqrt{3}/2)},\ E_{EDS,2} &= E_0'[1 + \pi\rho e^{\pi\rho}(1 - \sqrt{3}/2)]^{-1}, \\ G_{EDS} &= G_0[1 + \pi\rho e^{\pi\rho}(1-\nu_0)]^{-1}. \end{aligned} \tag{8}$$

NUMERICAL EXPERIMENTS

For our numerical simulations, we use the *Rotated Staggered Grid* FD scheme, which allows an accurate simulation of wave-propagation in media with high contrasts in elastic properties (Saenger et al., 2000).

Typical models have 2300×1000 ($y \times x$) gridpoints with periodic boundary conditions in the x-direction. Spacing between gridpoints in both directions is 0.0001 m. The cracked region is placed between a depth of 650 and 1650 gridpoints. For the elastic parameters of the homogenous background, we chose velocity $v_P = 5100$ m/s, velocity $v_{SV} = 2944$ m/s and the gravitational density $\rho_g = 2590$ kg/m^3. For cracks, we set $v_P = v_{SV} = 0$ and $\rho_g = 0.0001$ kg/m^3, which approximate vacuum. The source wavelet of our plane wave was always the first derivative of a Gaussian with a dominant frequency of 50 kHz and a time increment of 5×10^{-9} s. Two lines of geophones above and below the cracked region allow to measure the time delay of the mean peak amplitude caused by the cracked region and to calculate effective velocities. Due to the sharp contrasts in the model, computations were performed with second order spatial FD operators. Further details on such experiments can be found in Saenger and Shapiro (2002). From the obtained velocities, the P-wave modulus M and the shear modulus G are derived:

$$M = G(4G - E)(3G - E)^{-1} = \rho_g v_P^2, \quad G = \rho_g v_{SV}^2. \tag{9}$$

Randomly distributed parallel cracks

The models we examined in this section contained randomly distributed cracks parallel to the x-axis (see Fig. 1 left). We compared our results for different crack densities (Tab. 1) to the predictions of the theories described in section 2. This comparison is given in Fig. 2. The DS theory is in good agreement with our results, while the NIA and the EDS theory are not. Similar results for isotropic crack distributions were presented in Saenger and Shapiro (2002).

No.	crack density ρ	length of cracks [0.0001m]	number of cracks	porosity ϕ of the crack region	number of model realizations
1p	0.025	56	32	0.0018	1
2p	0.050	56	64	0.0036	1
3p	0.100	56	128	0.0073	5
4p	0.200	56	255	0.0145	5
5p	0.400	61	430	0.0262	1
1f	0.1	61	107	0.0098	3
2f	0.2	61	215	0.0193	3
3f	0.3	61	322	0.0291	3
4f	0.4	61	430	0.0388	3
5f	0.6	61	644	0.0581	3
6f	0.8	61	860	0.0775	3

Table 1: Details of crack arrangements. Model numbers with "p" refer to randomly distributed cracks parallel to the x-axis, those with "f" refer to two families of parallel cracks.

Usually, effective media theories rely on the assumption of large numbers of cracks. To examine the importance of this factor, we simulate media with only 20 parallel cracks ($\rho = 0.018$). The scattering of our results for different model realizations turnes out to be negligible. Yet, if ρ increases while the number of cracks remains constant, this scattering becomes more significant. To study this problem by the means of simulating wave propagation however becomes difficult as the numerical setup runs out of the long wavelength limit.

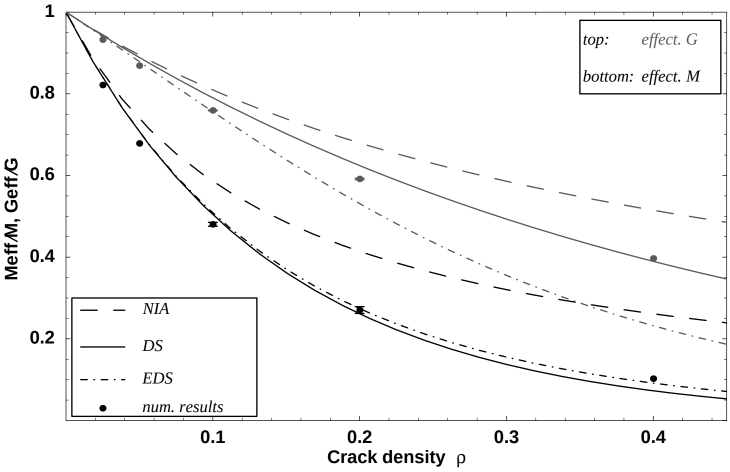

Figure 2: Normalised effective moduli for parallel cracks versus crack density. Dots: Numerical results, Lines: Theoretical predictions taken from the Non-interaction approximation (NIA), the Differential scheme (DS) and an extension of the DS (EDS).

Two families of parallel cracks

The arrangements we studied in this section contained two families of randomly distributed parallel cracks inclined at $30°$ to each other (see Fig. 1 middle). Both families have the same crack density. For details, see Table 1.

In our simulations, the sensitivity of velocities to crack density turned out to be much more significant for P-waves than for S-waves. Therefore we focus here on P-waves. Fig. 3 shows our results from simulations for different crack densities. Both the NIA and the EDS differ from our data for the fast symmetry direction. For the slow symmetry direction, our data matches the EDS predictions quite well, while the NIA is not satisfying. The behaviour of the EDS theory may be a consequence of its heuristic derivation.

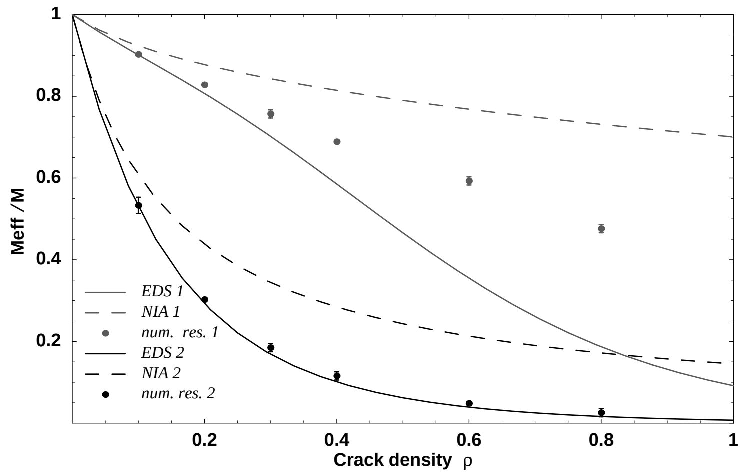

Figure 3: Numerical results (dots) for wave propagation in media with two families of parallel cracks along the “fast” (1) and the “slow” (2) principal axes. The curves show theoretical predictions taken from NIA and EDS.

IMPACT OF CRACK CLUSTERING ON EFFECTIVE PROPERTIES

In this section we study arrangements containing columnar clusters of five parallel cracks. The locations of these clusters are at random (see Fig. 1). We performed numerical simulations for different crack densities, where for each crack density the vertical spacing v was varied in order to examine the effect of shielding interactions (details of our models are displayed in Tab. 2). These interactions are described by Kachanov (1993). The dimensions, elastic parameters and experimental setup for these models are the same as for the models in the previous section. Yet, as source wavelet we used the first derivative of a Gaussian with a dominant frequency of 85 kHz and a time increment of 8×10^{-9} s.

Model	Crack- Density	Crack- Length [m]	Crack- Number	Porosity Φ	Vertical Spacing v $[10^{-4}$m]		
1r	0.1	0.0061	112	0.0068			
1.1b-1.3b	0.1	0.0061	112	0.0068	5	15	50
2r	0.2	0.0061	215	0.0131			
2.1b-2.3b	0.2	0.0061	215	0.0131	5	15	50
3r	0.4	0.0061	430	0.0262			
3.1b	0.4	0.0061	430	0.0262	5	15	

Table 2: Details of the crack arrangements (cluster configuration). Model numbers with “r” refer to random, those with “b” refer to a block model. For vertical spacing v, refer to Fig. 1.

Some general observations can be made (see Fig. 4): For all columnar models, P-wave velocity decreases less than for the random model with the same crack density. This means that the shielding effects of crack interactions dominate over the amplifying effects for this particular arrangement. The shielding effects gain importance as crack density increases. This trend is less clear for SV-wave velocities. These observations agree with the predictions of Kachanov (1993).

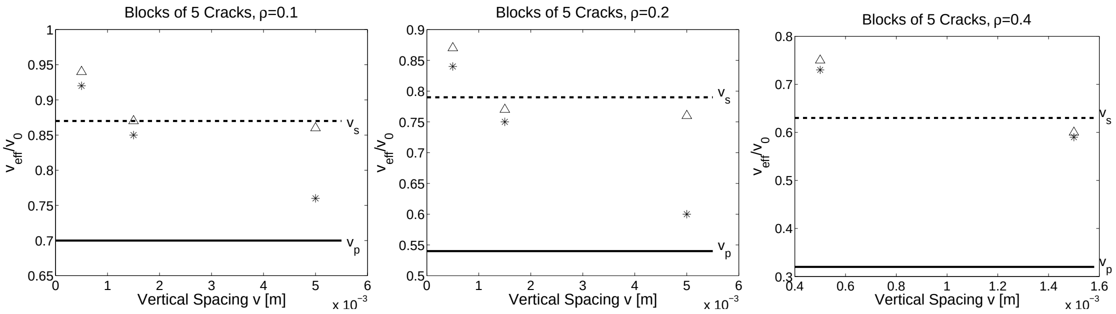

Figure 4: Numerical results (stars: P-, triangles: SV-waves) for $\rho = 0.1$ (left), $\rho = 0.2$ (middle) and $\rho = 0.4$ (right). Lines (solid: P-, dashed: SV-waves) indicate normalised velocities for random models.

CONCLUSIONS

The Rotated Staggered Grid FD scheme allows fast and accurate modelling of elastic wave propagation in fractured media. We concentrate on two dimensional models (plane strain case) which contain different distributions of parallel cracks. Effective elastic properties are determined in well controlled numerical experiments. We compare them to theoretical predictions for the microstructures we studied.

One series of simulations was concerned with randomly distributed parallel cracks. The match between predictions given by different effective media theories and our numerical results was best for the DS and for the EDS theory. For the second series we considered models with two families of randomly distributed parallel cracks at an angle of $30°$ to each other. Only theories like the NIA and the EDS theory, which take crack orientation statistics into account explicitely, can be applied here. The predictions by the NIA do not agree with our data satisfyingly, while agreement with the EDS theory is partly acceptable.

Finally, we simulated columnar clusters of parallel cracks and varied spacing between cracks within the clusters. We could confirm qualitative predictions given by Kachanov (1993). According to our results, clusters of cracks can result in significantly stiffer elastic moduli than randomly distributed cracks do. This clearly underlines the need for any effective media theory to take the statistics of crack centres into account, if the latter are not random.

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