CONVERGENCE OF THE TRAVELTIME POWER SERIES FOR A LAYERED TRANSVERSELY ISOTROPIC MEDIUM

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ABSTRACT

In the classical literature, seismic reflection traveltimes layers were considered for elastic isotropic layers. For long-offset, multi-component data and converted waves, the effects of anisotropy cannot be neglected. As a consequence, the consideration of transversely isotropic layered media with a vertical axis of symmetry, has become a topic of high interest. For both isotropic and anisotropic layers, the traveltime and offset for a multiply transmitted and reflected wave are expressed as parametric functions of horizontal slowness, which are analytic within some disc centered at the origin.
An explicit, closed-form expression for traveltime against offset is impossible, but one can determine a representation of it as a power series. We show that for SH and qP waves, such a power series always exists for small offsets. For qSV propagation in one or more layers, the power series does not exist when the NMO-velocity squared is zero. For a qSV reflection in a single layer this corresponds to an incipient triplication. For reflections in VTI media, no bounds for the radius of convergence has been established. We review and comment the fact that, for isotropic layered media, lower and upper bounds for the radius of convergence have been obtained. The power series for traveltime squared is always convergent within the radius of covergence for traveltime. No improved bound for the radius of convergence for traveltime squared seems to exist.
When there is a triplication in traveltime for a qP-qSV or qSV wave, the power series for traveltime is, at the most, valid up to the point where the traveltime starts to backtrack. This is illustrated in a numerical example for an on-axis triplication for reflected qSV waves. Close to incipient triplication, the offset range of validity of common traveltime approximations is very limited.

INTRODUCTION

To a large extent, routine seismic processing is based on traveltime expressions that are computed for a stack of homogeneous plane layers. The standard common-midpoint (CMP) method uses the so-called normal moveout for P-waves in isotropic horizontally stratified media Dix (1955). The normal moveout is nothing else than the second-order Taylor expansion of traveltime squared in such media.

Many studies of the P-wave isotropic power series of traveltime (or traveltime squared) against offset are reported in the literature Slotnick (1959); Taner and Koehler (1969a); Brown (1969); Al-Chalabi (1973); Ursin (1977); Hubral and Krey (1980); Castle (1994). All these investigations tacitly assume the existence of some least radius of convergence. In other words, for sufficiently small offsets, the power series converges and represents the traveltime. Papers that explicitly address the problem of existence and also some estimation of the radius of convergence are much more seldom. Goldin (1986) shows that the power series for traveltime is always convergent for “sufficiently smallt’t’ offsets. However, it cannot have radius of convergence larger that the sum of total distance traversed along the ray path. Specific classes of velocity distributions in which the radius of convergence meets that sharp upper bound are also described. Tygel (1994) considers arbitrary velocity distributions and provides a (non-sharp), model-dependent lower bound for the radius of convergence. Both authors restrict their investigations to traveltime. Sharper results for traveltime squared, as conjectured, have not been obtained sofar.

More recent investigations have shown that in many situations better processing results can be obtained by considerating anisotropy within the layers. The most popular case is that of transversely isotropy with vertical symmetry axis, simply referred to as VTI media Thomsen (1986). Traveltimes within VTI horizontally layered media, although certainly more complex, exhibit many similarities with their corresponding isotropic counterparts (see Thomsen, 1986; , Tsvankin and Thomsen, 1994; , Stovas and Ursin, 2003; , Ursin and Stovas, 2005; and other references therein).

The possible triplications of a qSV group velocity function represents a new complication Thomsen and Dellinger (2003). Dellinger (1991) has shown that triplication cannot occur for qP- and SH-waves. It can occur for qSV-waves propagating at three different angles with respect to the axis of symmetry: parallel to the symmetry axis, perpendicular to the symmetry axis, or off-axis, at an angle in between. For a reflected wave only the on-axis and off-axis triplications can occur. Conditions on the elastic constants, under which triplications can occur, are given by Thomsen and Dellinger (2003). We shall consider geometric traveltimes only, and not consider the complex wave for offsets associated with triplications at finite frequency Burridge (1967).

Traveltime and offset for a multiply transmitted and reflected ray within a layered VTI media are given by analogous expressions as the ones for isotropic media, with the fundamental exception that group quantities (velocity and angle) replace their corresponding phase counterparts. After transformation to phase velocity and angle, the VTI traveltime and offset become more complicated (although still analogous to isotropic) parametric expressions of horizontal slowness or ray parameter. Explicit elimination of the ray parameter is again not possible, so that one must resort to a power-series expansion to represent traveltime as a function of offset.

In Appendix A it is proven that such a power series exists when the squared NMO-velocity is different from zero. The NMO-velocity squared can only be zero for a qSV-wave or converted qP-qSV waves. For a reflected qSV-wave in a homogeneous layer, this corresponds to an incipient triplication, when the ray angle pauses in its forward motion but does not backtrack as slowness increases. The proof only shows the existence of a power series, but no estimate of its radius of convergence is given.

In Appendix B, we review and comment the convergence results that exist for the isotropic case. In the case of a homogeneous single layer, the normal moveout exactly represents, for all offsets, the traveltime squared of a non-converted reflection of the planar bottom interface. This means that, in this simple situation, the traveltime squared, as a function of offset, is given by a power-series of infinite radius. This favorable result does not hold if we consider traveltime instead of traveltime squared. In that case, the traveltime, as the square-root of the normal moveout, is represented by a power series that is convergent for offsets less than twice the reflector depth only. The much better convergence of traveltime squared than traveltime as a function of offset in the case of a single layer, lead geophysicists to conjecture that the same behavior should also hold for multiply reflected and transmitted waves within a stack of layers.

For two-terms power series, traveltime is approximated by a parabolic function, and traveltime squared is approximated by a hyperbolic function. Ursin (1977) has shown that, for a stack of isotropic layers, the standard hyperbolic approximation has less error. For higher-order approximations, no proof exists, but numerical evidence indicates that traveltime squared gives a better approximation, also for VTI layers.

In the same way as in the isotropic case, the VTI power series traveltime as a function of offset (truncated in its first two or three terms) is used for a variety of seismic processing purposes, including, e.g., velocity analysis Alkhalifah (1997a) and geometric-spreading correction Ursin and Hokstad (2003). In particular, first power-series coefficients are important to design alternative traveltime functions of offset. This is the case of non-hyperbolic traveltimes, shown to better approximate reflection events, especially for large offsets (see Alkalifah, 1997b and Ursin and Stovas, 2005).

The accuracy of the traveltime approximations obtained with the truncated Taylor series is investigated for qSV reflected waves in a few models near the incipient triplication point. The traveltime approximations have a very small useful offset range in this case, confirming the results of Ursin and Stovas (2005), all traveltime approximations break down at an off-axis triplication of a qSV-wave.

TRAVELTIME AND OFFSET FUNCTIONS OF SLOWNESS

We consider wave propagation in a stack of horizontal homogeneous VTI layers. For a multiple transmitted and reflected SH-wave or multiple transmitted, reflected and converted qP-qSV-wave, the traveltime, t, and offset, x, are governed by the expressions Ursin and Stovas (2005),

$$t(p) = \sum_k \frac{\Delta z_k}{V_k \cos\alpha_k} = \sum_k \frac{\Delta z_k}{v_k \cos\theta_k}\left(1 + p\frac{v_k'}{v_k}\right), \tag{1}$$

and

$$x(p) = \sum_k \Delta z_k \tan\alpha_k = \sum_k \frac{v_k \Delta z_k}{\cos\theta_k}\left(p + \frac{v_k'}{v_k^3}\right). \tag{2}$$

The index k represents a summation along the ray, and the quantities in the sums are computed for the proper wave mode; qP-, qSV- or SH-wave. In the above equations, for each layer k, Δz_k represents the thickness, V_k and v_k the group and phase velocities and α_k and θ_k are the group and phase angles, respectively. Moreover, v_k' denotes the p-derivative of v_k. Finally, we have used the invariance (Snell's law) of horizontal slowness or ray parameter

$$p = \sin\theta_k / v_k, \tag{3}$$

for all layers k, which is valid for a horizontally stratified medium.

The above expressions of traveltime and offset can be recast into a more convenient form by introducing the vertical slowness

$$q_k = \sqrt{1/v_k^2 - p^2} = \cos\theta_k / v_k \,. \tag{4}$$

Substituting into equations (1) and (2) yields

$$t(p) = \sum_k \Delta z_k (q_k - p\, q_k') \quad \text{and} \quad x(p) = -\sum_k \Delta z_k\, q_k'. \tag{5}$$

where $q_k' = dq_k/dp$, and we have used

$$\frac{v_k'}{v_k^3} = -\frac{1}{2}\left(\frac{1}{v_k^2}\right)' = q_k\, q_k' + p. \tag{6}$$

It is seen that the vertical slowness functions $q_k(p)$ determines the behavior of $t(p)$ and $x(p)$.

Expanding $q_k(p)$ in Taylor series and interchanging the order of summation gives Ursin and Stovas (2005)

$$t(p) = t(0) + \frac{1}{2} t(0) v_{NMO}^2\, p^2 + \frac{3}{8}\, t(0)\mu_4 p^4 + \cdots\ , \tag{7}$$

and

$$x(p) = t(0) v_{NMO}^2\, p + \frac{1}{2}\, t(0)\mu_4\, p^3 + \cdots\ , \tag{8}$$

where

$$t(0) = \sum_k \frac{\Delta z_k}{v_{0,k}} = \sum_k \Delta t_{0,k}, \quad \text{and} \quad v_{NMO}^2 = \frac{1}{t(0)} \sum_k v_{0,k}^2 a_{0,k} \Delta t_{0,k}, \tag{9}$$

are the vertical traveltime and NMO-velocity, repectively. In the above equations, $v_{0,k}$ is the vertical velocity in layer k. Omitting the index k, that velocity is given by

$$\alpha_0 = \sqrt{C_{33}/\rho} \quad \text{or} \quad \beta_0 = \sqrt{C_{44}/\rho}, \tag{10}$$

for a qP-wave or qSV/SH-wave, respectively. We shall use the Thomsen parameters

$$\delta = \frac{(c_{13}+c_{44})^2 - (c_{33}-c_{44})^2}{2c_{33}(c_{33}-c_{44})}, \quad \epsilon = \frac{c_{11}-c_{33}}{2c_{33}}, \quad \text{and} \quad \gamma = \frac{c_{66}-c_{44}}{2c_{44}}. \tag{11}$$

It is also convenient to introduce the parameter σ defined by

$$\sigma = (\alpha_0/\beta_0)^2(\epsilon - \delta). \tag{12}$$

Ommitting the index k, the coefficient, a_0, defining the NMO-velocity squared in equation (9) is given by

$$a_0 = \begin{cases} 1 + 2\delta, & (\mathrm{qP-wave}), \\ 1 + 2\sigma, & (\mathrm{qSV-wave}), \\ 1 + 2\gamma, & (\mathrm{SH-wave}). \end{cases} \tag{13}$$

Finally, the coefficient, μ_4, is defined in Ursin and Stovas (2005),

$$\mu_4 = \frac{1}{t_0}\sum_k v_{0,k}^4[a_{0,k}^2 + 4a_{1,k}], \tag{14}$$

where, also ommitting the index k, the coefficient a_1 reads

$$a_1 = \begin{cases} 2(\epsilon - \delta)\left(1 + \dfrac{2\delta\alpha_0^2}{\alpha_0^2 - \beta_0^2}\right), & (\mathrm{qP-wave}), \\ -2\sigma\left(1 + \dfrac{2\delta\alpha_0^2}{\alpha_0^2 - \beta_0^2}\right), & (\mathrm{qSV-wave}), \\ 0, & (\mathrm{SH-wave}). \end{cases} \tag{15}$$

TRAVELTIME AS A FUNCTION OF OFFSET

Equations (5) express traveltime and offset as parametric functions of slowness. Elimination of that parameter will provide traveltime as a function of offset. That function will be denoted by $T(x)$. As a direct elimination of p is not possible, the common practice is to express $T(x)$ as a Taylor series around zero offset. In Appendix A it is shown that traveltime, $t(p)$, and offset, $x(p)$, are analytic functions of horizontal slowness within a disc $|p| < p_m$. Furthermore, when

$$x'(0) = t(0)v_{NMO}^2 \neq 0, \tag{16}$$

the function $x(p)$ admits an analytic inverse, $p(x)$, defined within a sufficiently small disc, $|x| < r_c$, centered at the origin of the complex x-plane. This we can substitute into the traveltime function, $t(p)$, to obtain the composite function $T(x) = t(p(x))$. Due to the analyticity of both $t(p)$ and $p(x)$, as well as to the fact that $p(x)$ maps the disc $|x| < r_c$ into the domain of analyticity, $|p| < p_m$ of $t(p)$, it follows that $T(x)$ is a well defined, analytic function within $|x| < r_c$. As a consequence, $T(x)$ can then be represented by a convergent power series around the origin for offsets $|x| < r_c$, as desired. Because of reciprocity, that series, if it exists, has only even powers of x. From equations (7) and (8) it can be shown that

$$T(x) = T(0) + \frac{x^2}{2T(0)v_{NMO}^2} - \frac{\mu_4\, x^4}{8T(0)^3(v_{NMO}^2)^4} + \cdots \,. \tag{17}$$

Note that $T(0) = t(0)$ is the zero-offset traveltime. Higher-order terms can be found in Brown (1969) or Ursin and Stovas (2005). We see that if $v_{NMO}^2 = 0$, so that that condition (16) is violated, the power series $T(x)$ does not exist.

For a reflection from a single layer with no mode conversions, equation (9) gives $v_{NMO}^2 = v_0^2 a_0$. This is possibly zero or negative for a qSV-wave only Dellinger (1991). Then

$$v_{NMO}^2 = \beta_0^2(1 + 2\sigma), \tag{18}$$

which is zero for $\sigma = -0.5$. This is the condition for incipient on-axis triplication. Then the power series does not exist. For $\sigma < -0.5$ there is an on-axis triplication. In this situation the power series exists, but the range of convergence must be less than the offsets where the group velocity and traveltime start to backtrack.

For a stack of layers, the squared NMO-velocity can be zero or negative only if the wave has passed through one or more layers as a qSV-wave. The radius of convergence must be limited by the offset where traveltime starts to backtrack, either in an on-axis or an off-axis triplication. In both cases, this occurs when

$$x'(p) = -\sum_k \Delta z_k q_k'' = 0, \tag{19}$$

where equation (5) has been used.

The coefficients in the power series for $T(x)$ in equation (17) become large when v_{NMO}^2 is small (positive or negative), and we expect that the radius of convergence will be small.

TRAVELTIME SQUARED

The power series for traveltime squared can also be obtained from equations (7) and (8) Taner and Koehler (1969b); Hubral and Krey (1980); Ursin and Stovas (2005). In order to obtain the first terms, it is easier to square equation (17) which gives

$$T(x)^2 = T(0)^2 + \frac{x^2}{v_{NMO}^2} - \left[\frac{\mu_4}{(v_{NMO}^2)^2} - 1\right] \frac{x^4}{4T(0)^2(v_{NMO}^2)^2} + \cdots . \tag{20}$$

As discussed earlier, the convergence of the power series for traveltime carries over to the corresponding power series for traveltime squared. This means that, at each offset, x, for which $T(x)$ has a convergent power series, the same happens with the power series representation of $T^2(x)$, so that that the convergence of the power series for $T^2(x)$ is *at least* equal to the one for $T(x)$, but it can be larger. A striking example of the fact that traveltime squared may have better convergence properties (that is a larger radius of convergence) as compared to traveltime, is provided by the reflection from a single layer, as discussed in Appendix B. Then the radius of convergence for $T(x)$ is twice the layer thickness, while the series for $T^2(x)$ gives the exact expression with two terms. The terms for x^4 and higher are all zero. Numerical evidence suggests that taking the square root of the truncated series $T(x)^2$ gives more accurate traveltime approximation than the series $T(x)$ truncated to the same order in x^2.

NUMERICAL RESULTS

The behavior and accuracy of the truncated Taylor series for traveltime and the square root of traveltime squared were studied for simple two-layer models consisting of medium I with negative squared NMO-velocity and medium II with positive squared NMO-velocity for reflected qSV waves. The elastic parameters of the two media, as given in Table 1, are identical except for the slightly different values of ϵ.

Parameters	α_0 [km/s]	β_0 [km/s]	γ [km/s]	δ	σ
Medium I	2.0	1.0	-0.1	0.05	-0.6
Medium II	2.0	1.0	-0.05	0.05	-0.4

Table 1: Parameters of the medium I and II used in the modeling.

Five different models, all with total thickness equal to 1000 m, are described in Table 2. Model A is medium I with $v_{NMO}^2 = -0.2$, model B is 750 m medium over 250 m medium II, model C is 250 m medium I over 750 m medium II and model D is medium II. Finally, there is a model with $v_{NMO}^2 = 0$, consisting of 500 m medium I over 500 m medium II.

Model	A	B	C	D
I	100%	75%	25%	0
Medium II	0	25%	75%	100%
v_{NMO}^2 [km^2/s^2]	-0.2	-0.1	0.1	0.2
μ_4^2 [km^2/s^2]	5.48	5.03	4.12	3.67

Table 2: Model composition and traveltime parameters.

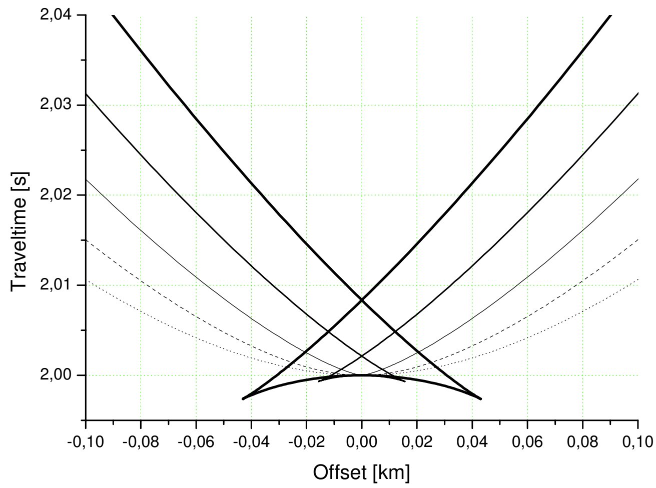

Figure 1: Traveltime as function of offset.

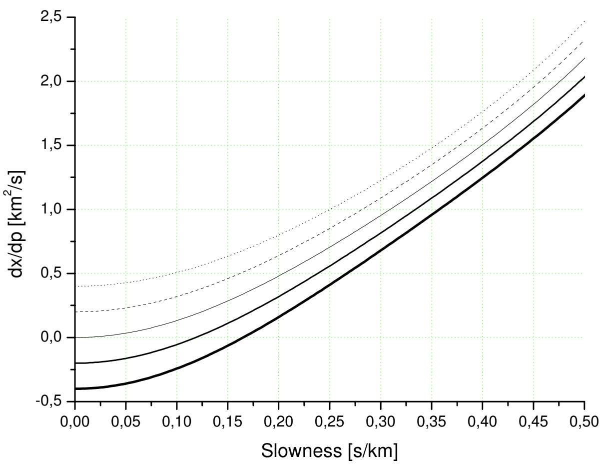

Figure 2: Derivative of slowness with respect to offset.

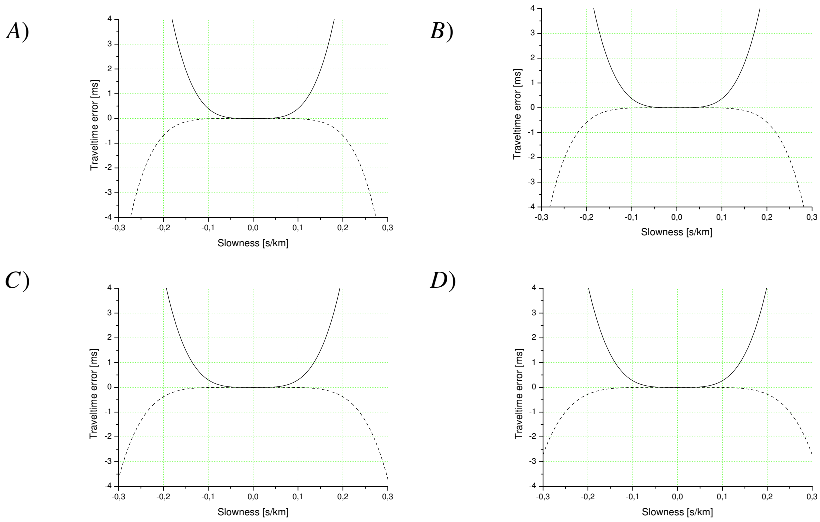

Figure 3: Errors of traveltime as function of slowness for second-order (solid line) and fourth-order (dashed line) approximations. The labels A-D refer to the models A-D in the text.

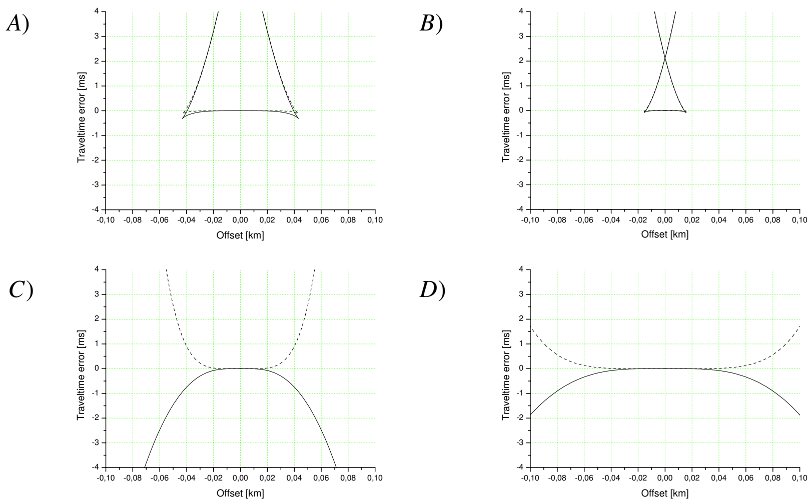

Figure 4: Errors of traveltime as function of offset for second-order (solid line) and fourth-order (dashed line) approximations. The labels A-D refer to the models A-D in the text.

Figure 1 shows traveltime computed as function of offset for the five models.

Figure 2 shows the derivative dx/dp for the same models. Triplication and backtracking occurs when $dx/dp < 0$. This happens for models A and B, with the former has the largest triplication domain. In the middle curve is shown the model with $v_{NMO}^2 = 0$. This results in a kink in the traveltime curve. For models C and D, $dx/dp > 0$ and there is no triplication.

Figure 3 shows the traveltime series for the Taylor series $t(p)$ in equation (7) with two and three terms. For the two models with triplication, the error is of the same sign, while for the two models with positive v_{NMO}^2, the error changes sign. In all cases, three terms give a more accurate approximation than two terms.

Figure 4 shows the errors of traveltime approximation $T(x)$ in equation (17) truncated after two and three terms. As expected, these approximations are only valid for offsets less than where the traveltime curve starts to backtrack (at $dx/dp = 0$). The approximations obtained by taking the square root of the truncated series for $T(x)^2$ in equation (20) are extremely similar, and the plots look almost identical to the ones in Figure 4. It is seen that the range of validity of the approximations for $T(x)$ for a qSV-reflection is very limited when $|v_{NMO}^2|$ is small. For a reflector at 1000 m depth, the error becomes larger than 4 ms for offsets larger than 20 - 100 m, depending on the model.

CONCLUSIONS

For a multiply transmitted and reflected wave in VTI media, traveltime and offset are expressed as parametric functions of horizontal slowness. Elimination of this parameter to provide traveltime (or traveltime squared) as a function of offset is in general not possible. We have shown that traveltime can be represented as a convergent power series around the origin for sufficiently small offsets, except in the case of a vanishing NMO-velocity. The NMO-velocity can only be zero for qSV propagation in some layer. When the squared NMO-velocity becomes negative there is an on-axis triplication in the traveltime, and the range of validity for the power series for traveltime is limited to the first branch of the traveltime function.

In the case of isotropic layered media the power series always exists, moreover, lower and upper bounds for the radius convergence were given. For both VTI and isotropic layers, the region of convergence of the power series expansion of traveltime squared always contains the corresponding convergence region of traveltime.

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APPENDIX A

ANALYTICITY OF FUNCTIONS

To show the analyticity of traveltime, $t(p)$, and offset, $x(p)$, as functions of horizontal slowness, p, it suffices to restrict the analysis to their single-layer contributions, $\Delta t(p)$ and $\Delta x(p)$ given by

$$\Delta t(p) = (q - pq')\Delta z \quad \text{and} \quad \Delta x(p) = -q'\,\Delta z, \tag{1}$$

in which, for convenience, we dropped the layer index, k. The vertical slowness squared is, for SH-waves,

$$q^2 = 1/\beta_0^2 - (1+2\gamma)p^2. \tag{2}$$

For qP- and qSV-waves, the corresponding vertical slowness function gets more complicated, being given by (see, e.g., Ursin and Stovas, 2005 , their equation (B-1))

$$q^2 = \frac{1}{2}\left(\frac{1}{\alpha_0^2} + \frac{1}{\beta_0^2}\right) - p^2(1+\sigma+\delta) \mp \frac{1}{2}\left(\frac{1}{\beta_0^2} - \frac{1}{\alpha_0^2}\right)\sqrt{1+bp^2+cp^4}, \tag{3}$$

with the $\mp$ sign meaning that q is associated with the qP- or qSV-velocity, respectively, and

$$b = -\frac{4\alpha_0^2\beta_0^2}{\alpha_0^2 - \beta_0^2}(\sigma - \delta) \quad \text{and} \quad c = \frac{4\alpha_0^4\beta_0^4}{(\alpha_0^2 - \beta_0^2)^2}\left[\frac{2(\alpha_0^2 - \beta_0^2)}{\alpha_0^2}\,\sigma + (\sigma + \delta)^2\right]. \tag{4}$$

Analyticity of traveltime and offset

From basic results of the theory of complex variables, we see that the function $q = q(p)$ possesses only branch-point singularities located at the zeros of the inner square root function $g(p) = \sqrt{1 + bp^2 + cp^4}$ and the zeros of the function $q(p)$ itself. Moreover, since $g(0)$ and $q(0)$ are (real) non-zero numbers (in fact, $g(0) = 1$ and $q(0) = 1/\alpha_0$ or $q(0) = 1/\beta_0$, depending on the sign choice $\mp$, respectively) we find that $q(p)$ is a well defined and analytic function within the disc $|p| < p_r$, where p_r is the smallest of the distances of its singularities with respect to the origin $p = 0$. Setting p_m to be the least of all single-layer radii $p_r = p_r^{(k)}$, we find that $t(p)$ and $x(p)$ are analytic at least within the disc $|p| < p_m$.

Analytic inverse of offset

We next investigate the existence of an analytic inverse of $x(p)$, defined within a sufficiently small disc centered at the origin of the complex x-plane. From the theory of complex variables (see, e.g., Churchill, 1960), the existence of such inverse function is guaranteed whenever $x(p)$ has a non-vanishing derivative at $p = 0$.

From equation (8) it is seen directly that

$$x'(0) = t(0)v_{NMO}^2\,. \tag{5}$$

Thus, the analytic inverse of $x(p)$, in a small disc centered at the origin, exists whenever $v_{NMO}^2 \neq 0$. This is always the case for SH- and qP-waves.

For a single layer, the power series exists when

$$\frac{d(\Delta x)}{dp} = v_0\,\Delta z\,a_0 = \begin{cases} \alpha_0\Delta z(1 + 2\delta) & (\mathrm{qP-wave}), \\ \beta_0\Delta z(1 + 2\sigma) & (\mathrm{qVS-wave}), \\ \alpha_0\Delta z(1 + 2\gamma) & (\mathrm{SH-wave}). \end{cases} \tag{6}$$

is different from zero. The expression becomes zero for a qSV-wave when $\sigma = -0.5$. This is the condition for “incipient triplication”, as discussed in Dellinger (1991) and Thomsen and Dellinger (2003). For $\sigma > -0.5$ there is no triplication of the qSV-wave, and for $\sigma < -0.5$ it exists.

APPENDIX B

LAYERED ISOTROPIC MEDIUM

When the layers are isotropic, the previous derivations simplify considerably Goldin (1986); Tygel (1994). In that case the slowness function is again defined by

$$q_k = \sqrt{1/v_k^2 - p^2}, \tag{1}$$

but now the velocity v_k is constant. Then $q_k' = -p/q_k$ and

$$t(p) = \sum_k \frac{\Delta z_k}{v_k^2 q_k} \quad \text{and} \quad x(p) = p\sum_k \frac{\Delta z_k}{q_k}\;. \tag{2}$$

The function $q_k(p)$ is analytic when $|p| < 1/v_k$, so that $t(p)$ and $x(p)$ are analytic in the disc

$$|p| < 1/v_{max}, \tag{3}$$

where v_{max} is the maximum velocity encountered along the ray. As before, $x(p)$ has an analytic inverse in the vicinity of the origin, $x(0) = 0$, when

$$t(0)v_{NMO}^2 = \sum_k v_k\Delta z_k \neq 0, \tag{4}$$

which is trivially the case. This concludes the proof of existence of a power series, $T(x)$, with a certain radius of convergence, r_c, around $x = 0$. Goldin (1986) has obtained an upper bound, and Tygel (1994) has obtained a lower bound for this radius. Their results can be combined in the form

$$\sum_k \frac{\Delta z_k}{\sqrt{1+(v_{max}/v_k)^2}} \leq r_c \leq \sum_k \Delta z_k \ . \tag{5}$$

For a reflected ray from a single interface,

$$T(x) = \frac{2\Delta z}{v}\sqrt{1+\left(\frac{x}{2\Delta z}\right)^2}, \tag{6}$$

where v and Δz are the velocity and thickness of the single layer. The square root can be expanded in a convergent power series when $|x| < 2\Delta z$, which corresponds exactly to the upper bound in equation (5). The expression for $T(x)^2$ is exact in this case.