ZERO-OFFSET SEISMIC AMPLITUDE DECOMPOSITION AND MIGRATION

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ABSTRACT

Recent studies on the propagation matrices of the one-way normal and normal-incidence-point (NIP) waves in arbitrary anisotropic media have been shown to provide attractive decomposition expressions for (two-way) zero-offset PP and SS reflections. In particular, the geometrical spreading, KMAH index and transmission and reflection coefficients of the zero-offset ray could be simply expressed in terms of products or sums of the corresponding quantities of the one-way normal and NIP waves. The ray-theoretic Green's function for the reflected wave is equal to twice the product of the Green's function of the NIP wave and the normal wave. True-amplitude post-stack depth migration is done by reverse-propagating wavefield in a half-velocity model with double density and elastic parameters divided by two. An estimate of the normal-incidence reflection coefficient is obtained from the reverse-propagated wavefield at time equal zero, followed by an additional geometrical spreading correction.

INTRODUCTION

A stacked seismic section is considered as an approximation of a zero-offset seismic section. Each trace is the result of a seismic experiment where the source and receiver are located at the same surface point. This corresponds to a normal incidence ray which has been reflected at the normal incidence point (NIP) and then is returning to the source/receiver point at the same path. In anisotropic media this can occur when the slowness vector is parallel to the surface normal at the NIP interface. For a multiple transmitted, reflected and converted wave this requires that the ray code up is the ray code down in reverse order. This includes the PP and SS primary reflections, but not the PS converted wave reflection.

A stacked section is obtained by summing the traces in a common-midpoint gather after they have been corrected for normal moveout (NMO). This requires the knowledge of the NMO or stacking velocity which for 3-D data is given by a 2×2 symmetric matrix (Ursin, 1982). Krey (1976) and Chernyak and Gritsenko (1979) showed that this NMO velocity matrix can be computed from the curvature of a fictitious one-way wave that starts as a point source at the NIP reflection point. Such wave was called the *NIP wave* by Hubral (1983).

For the purpose of migration of stacked data, Loewenthal et al. (1976) introduced the exploding reflector model for zero-offset primary reflections. This is another fictitious experiment in a fictitious medium with half the velocity of the true medium. The sources are placed along the reflector, and all sources are activated at the same time set equal to zero. An equivalent description of the exploding reflector model is that, in the vicinity of each point of the reflector, a wave front with the same shape as the reflector starts its way up and is recorded at the surface. Such one-way waves have been called *normal waves* by Hubral (1983).

Iversen (2006) gave a complete review of normal incidence reflections with many new results. While Hubral (1983) considered wavefront curvature in an isotropic medium, Iversen (2006) considered Green's functions in anisotropic elastic media for PP and SS primary reflections expressed by asymptotic ray theory.

In particular, he obtained results for the relative geometrical spreading and the KMAH index for the normal-incidence reflected wave, the NIP wave and the normal wave. These results are valid for any normal-incidence reflected wave such that the ray code up is equal to the ray code down in reverse order. That is, the wave type in a layer for the wave up is the same as for the wave down. We simplify Iversen's results by using flux-normalized reflection and transmission coefficients instead of amplitude-normalized ones. Then the product of the reflection and transmission coefficients for the down-going wave is equal to the corresponding product for the up-going wave (Chapman, 1994). For each wave, we define a geometric ray approximation by a complex amplitude in the frequency domain by traveltime, KMAH index, relative geometrical spreading and a product of reflection and transmission coefficients. Then we show that the Green's function of the normal-incident reflected wave is equal to two times the product of the Green's functions of the NIP wave and the normal wave. For completeness, Appendix A contains a discussion on the a different geometrical spreading decomposition (Tygel et al., 1994; Schleicher et al., 2001).

Map migration (Kleyn, 1977) is the transformation of a two-way normal-incidence traveltime map into a reflecting interface at depth, given the smooth parameters of the elastic medium. Gjoystdal and Ursin (1981) extended this to also estimating the medium parameters, given non-zero offset traveltimes or stacking velocities. Gjoystdal et al. (1984) proposed to use the normal wave to downward continue the curvature of the zero-offset reflection to obtain the curvature of the interface. We may also estimate the true amplitude of the zero-offset reflection. By dynamic ray tracing along normal-incidence rays, we compute amplitude corrections which can be used to estimate the normal-incidence reflection coefficient at the depth interface.

Post-stack depth migration is normally done by downward continuing the recorded data in a model with half the medium velocity (Loewenthal, 1976; Gazdag snd Sguazzero, 1984). An estimate of the reflectivity is then obtained from the reverse-propagated wavefield at time equal zero. In Appendix B it is shown how to extend this scheme to anisotropic media by using elastic parameters divided by two and density multiplied by two. This will half the phase and group velocities of the medium. The traveltime is doubled, so that time equal zero in the new medium corresponds to half the traveltime in the data, exactly at the reflection point. This downward continuation corresponds to downward continuation of the normal wave, except for the doubling of the traveltime. Therefore there is an uncompensated part of the geometrical spreading, which remains in the amplitude of the migrated image. In order to estimate this residual geometrical spreading correction, we propose to do a second zero-offset Kirchhoff migration (Hubral et al., 1991) without geometrical spreading correction.

DECOMPOSITION OF NORMAL-INCIDENCE REFLECTIONS

We consider an anisotropic elastic medium with a multiple transmitted, reflected and converted wave which is reflected back at a point $\mathbf{y}$ at an interface with the slowness or phase velocity parallel to the interface normal, and such that the ray code for the upgoing wave is the wavecode for the downgoing wave in reverse order. Then the upgoing wave comes back to the source point $\mathbf{x}$ which is a zero-offset point, see Figure 1.

The geometric ray approximation for the reflected ray is, in the frequency domain, proportional to the scalar response function

$$U_R = \frac{e^{-i\frac{\pi}{2}\,\chi_R\,\mathrm{sgn}\omega}\;e^{i\omega T_R}}{\mathcal{L}_R}\,t_U\,r_N\,t_D, \tag{1}$$

where T_R is the two-way traveltime, χ_R is the KMAH index, and t_D and t_U are the product of the transmission and reflection coefficients for the wave going up and down, respectively. All the reflection and transmission coefficients are normalized with respect to the energy flux normal to each interface, and then the coefficients are reciprocal Chapman (1994), so that

$$t_D = \Pi t_{Dk} = \Pi t_{Uk} = t_U. \tag{2}$$

The normal-incidence reflection coefficient at the last interface is r_N.

The relative geometrical spreading is

$$\mathcal{L}_R = \left|\det \mathbf{Q}_2(\mathbf{x},\mathbf{y},\mathbf{x})\right|^{1/2}, \tag{3}$$

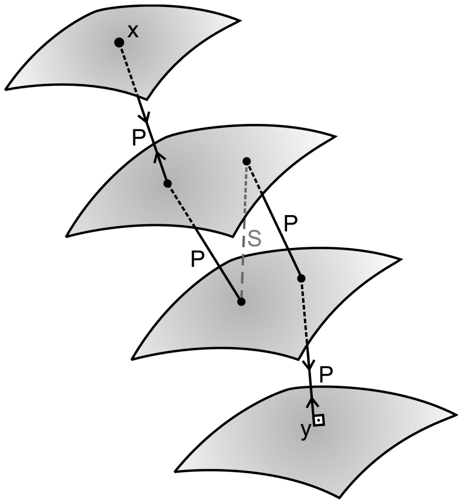

Figure 1: The ray path of a multiple transmitted, reflected and converted NIP path.

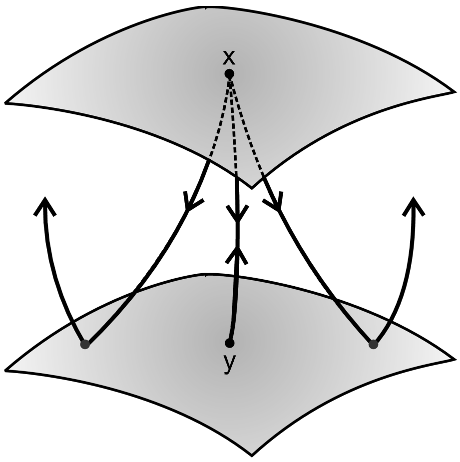

Figure 2: The reflected wave from a point source with areal receivers.

where the 2×2 matrix $\mathbf{Q}_2(\mathbf{x}, \mathbf{y}, \mathbf{x})$ is part of the two-way 4×4 ray propagator matrix $\mathbf{\Pi}(\mathbf{x}, \mathbf{y}, \mathbf{x})$. In terms of the one-way upward ray propagator matrix Červený (2001), Chapman (2004)

$$\mathbf{\Pi}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix}, \tag{4}$$

this can be expressed by (Iversen, 2006)

$$\mathbf{\Pi}(\mathbf{x}, \mathbf{y}, \mathbf{x}) = \begin{pmatrix} 2\mathbf{Q}_2(\mathbf{P}_1 + c^{-1}\mathbf{P}_2\,\mathbf{D})^{\mathbf{T}} + \mathbf{I} & 2\mathbf{Q}_2(\mathbf{Q}_1 + c^{-1}\mathbf{Q}_2\,\mathbf{D})^{\mathbf{T}} \\ 2\mathbf{P}_2(\mathbf{P}_1 + c^{-1}\mathbf{P}_2\,\mathbf{D})^{\mathbf{T}} & 2\mathbf{P}_2(\mathbf{Q}_1 + c^{-1}\mathbf{Q}_2 D)^T - \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I}^* & \mathbf{O} \\ \mathbf{O} & \mathbf{I}^* \end{pmatrix}, \tag{5}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{I}^* = \mathrm{diag}(1, -1)$ and $\mathbf{D}$ is the curvature matrix of the normal-incidence reflector, all of these being 2×2 matrices. The reflected wave is schematically shown in Figure 2. The NIP wave is a hypothetical wave that starts as a point source of unit amplitude at the normal-incidence point and propagates within the original medium towards the given zero-offset surface location where it is recorded, see Figure 3 (left). The response function is (see equation (1))

$$U_{NIP} = \frac{e^{-i\frac{\pi}{2}\,\chi_{NIP}\,\mathrm{sgn}\omega}\; e^{i\omega T_R/2}}{\mathcal{L}_{NIP}}\, t_U, \tag{6}$$

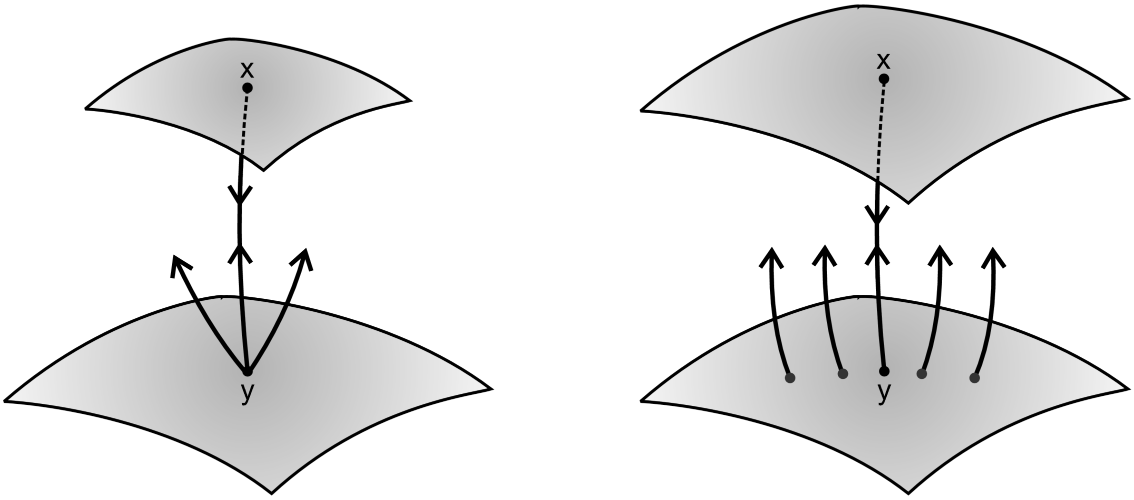

Figure 3: Left: The NIP wave starting at the normal incidence reflection point. Right: The normal wave starting at the interface.

where the relative geometrical spreading factor, $\mathcal{L}_{NIP} = |\det \mathbf{Q}_{NIP}|^{1/2}$ (see equation (3)), is computed from

$$\begin{pmatrix} \mathbf{Q}_{NIP} \\ \mathbf{P}_{NIP} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} \mathbf{O} \\ \mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_2 \\ \mathbf{P}_2 \end{pmatrix} , \tag{7}$$

and χ_{NIP} is the KMAH index for the NIP wave.

A second hypothetical wave is the normal wave, which which originates at the reflecting surface with a wavefront curvature of the interface at that point, see Figure 3 (right).

We shall let the normal wave start with amplitude r_N, the normal-incidence reflection coefficient, so that the response function is

$$U_N = \frac{e^{-i\frac{\pi}{2}\chi_N \,\mathrm{sgn}\omega} \; e^{i\omega T_R/2}}{\mathcal{L}_N} \, t_U \, r_N \, . \tag{8}$$

The geometrical spreading, $\mathcal{L}_N = |\det \mathbf{Q}_N|^{1/2}$ is computed from (see equation (7))

$$\begin{pmatrix} \mathbf{Q}_N \\ \mathbf{P}_N \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ v^{-1}\mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_1 + v^{-1}\mathbf{Q}_2\mathbf{D} \\ \mathbf{P}_1 + v^{-1}\mathbf{P}_2\mathbf{D} \end{pmatrix}, \tag{9}$$

where v is the phase velocity at the normal-incidence point, and χ_{NIP} is the KMAH index for the NIP wave. Combining equations (5), (7) and (9), gives (Iversen, 2006)

$$\mathbf{\Pi}(\mathbf{x},\mathbf{y},\mathbf{x}) = \begin{pmatrix} 2\mathbf{Q}_{NIP}\mathbf{Q}_N + \mathbf{I} & 2\mathbf{Q}_{NIP}\mathbf{P}_N \\ 2\mathbf{P}_{NIP}\mathbf{P}_N & 2\mathbf{P}_{NIP}\mathbf{Q}_N - \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I}^* & \mathbf{O} \\ \mathbf{O} & \mathbf{I}^* \end{pmatrix} . \tag{10}$$

From this, it follows that

$$\mathbf{Q}_2 = 2\mathbf{Q}_{NIP}\mathbf{P}_N\mathbf{I}^*, \tag{11}$$

so that

$$\mathcal{L}_R = 2\mathcal{L}_{NIP}\mathcal{L}_N. \tag{12}$$

Furthermore, Iversen (2006) has shown that

$$\chi_R = \chi_{NIP} + \chi_N. \tag{13}$$

Combining equations (1), (6) and (13) with the two equations above and the reciprocity relation for the upward and downward transmission coefficients (2), gives the decomposition formula

$$U_R = U_{NIP}\, U_N/2. \tag{14}$$

The factor 2 can be avoided by letting the NIP-wave have strength 2, but this has consequences for other relations.

TRUE AMPLITUDE DEPTH MIGRATION

We consider the depth migration of PP or SS primary reflections under the assumption of a smooth background model. Moreover, we also assume that multiples have been adequately attenuated or removed by pre-processing. Based on the above decomposition formula, we can devise the following two schemes for true-amplitude zero-offset migration. The first scheme consists of a map migration of selected reflectors (see, e.g., Kleyn, 1977, Gjoystdal and Ursin, 1981 and Gjoystdal et al., 1984) in which true amplitudes (i.e., amplitudes corrected for geometrical spreading) are attached. The second method is a full, true-amplitude poststack depth migration that consists of a cascaded application of downward continuation (see, e.g., Loewenthal, 1976) followed by and amplitude correction factor that is obtained by means of an unweighted Kirchhoff depth migration (see, e.g. Hubral et al., 1991 and Ursin, 2004). Following the lines of Schleicher et al. (2001), these procedures can be seen to be valid for anisotropic media. A quick algorithmic description of the two schemes is given below.

True-amplitude map migration: Estimate the two-way zero offset traveltime, $T_k(x^0, y^0)$, and the corresponding amplitudes, $A_k(x^0, y^0, t^0)$, for a selected number of horizons. For each horizon estimate the first derivatives and, possibly, second derivatives of traveltime. these are used to downward continue the normal wave to half the traveltime, $T_k/2$. This gives the reflecting interface at $z = \phi_k(x, y)$. Compute the KMAH index, $\chi_R = \chi_N + \chi_{NIP}$, and, if necessary, apply a phase correction to the data and improve the amplitude estimate. An estimate of the reflection coefficient, $r_N(x, y)$ is obtained by multiplying the reflection amplitude by the geometrical spreading in equation 12.

Wave-equation continuation: The recorded zero-offset seismic data, $u_R(\mathbf{x}^0, t)$, $\mathbf{x}^0 = (z^0, y^0, z^0)$, in which the recording surface needs not be planar, are reverse-propagated downward in a half-velocity model. In anisotropic media this is obtained by multiplying the density by two and dividing the elastic parameters by two (see Appendix B). This corresponds to reverse-propagating the normal wave, so that this part of the geometrical spreading for the reflected wave is compensated. The image at depth is taken for time equal zero. This is approximately

$$I_R = \frac{r_N}{2\mathcal{L}_{NIP}} \ . \tag{15}$$

In order to estimate and correct for the geometrical-spreading factor, $\mathcal{L}_{NIP}$, perform a Kirchhoff depth migration with unit weight on the original zero-offset data. Following Hubral et al. (1991), the resulting amplitude at depth is given by

$$I_K = \frac{r_N}{\mathcal{L}_{NIP}^2} \ . \tag{16}$$

Combining the last equations gives the correction factor

$$C = 2\,\mathcal{L}_{NIP} = 4\,\frac{I_R}{I_K} \ . \tag{17}$$

This must be smoothed before being applied to the data. The final result of the depth migration is an estimate of the normal-incidence reflection coefficient

$$r_N(\mathbf{x}) = u_R(\mathbf{x}, 0)C(\mathbf{x}) \ , \tag{18}$$

where $u(\mathbf{x}, 0)$, $\mathbf{x} = (x, y, z)$, is the reverse-propagated wavefield in the half-velocity model taken at time equal zero.

CONCLUSIONS

We have shown that the ray theoretical Green's function for the reflected wave is equal to twice the product of the Green's functions of the NIP wave and the normal wave. Using the normal wave resulted in a scheme for true-amplitude map migration. True-amplitude post-stack depth migration can be done by reverse-propagating the recorded wavefield in a model with half the elastic constants and double the density. This results in a half-velocity model in anisotropic media. The image is formed by taking the reverse-propagated wavefield at time equal zero followed by an additional geometrical spreading correction.

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APPENDIX A

CONNECTION WITH AN ALTERNATIVE GEOMETRICAL SPREADING DECOMPOSITION

In this appendix we relate the geometrical spreading decomposition provided by equation 12 with a decomposition provided in Schleicher et al. (2001), which is valid for an arbitrary finite-offset, primary-reflection ray in anisotropic media. For zero offset and in our notation, that decomposition reads (compare with equation (B.19) of Schleicher et al., 2001)

$$\mathcal{L}_R(\mathbf{x},\mathbf{y},\mathbf{x}) = \mathcal{L}(\mathbf{x},\mathbf{y})\mathcal{L}(\mathbf{y},\mathbf{x})\mathcal{L}_H(\mathbf{y}), \tag{A-1}$$

where $\mathcal{L}(\mathbf{x},\mathbf{y},\mathbf{x})$ is the point-source relative geometrical spreadings of the (two-way) zero-offset ray and $\mathcal{L}(\mathbf{x},\mathbf{y})$ and $\mathcal{L}(\mathbf{y},\mathbf{x})$ are the point-source relative geometrical spreading of the (one-way) down and up-going rays that connect the source-receiver location, $\mathbf{x}$ to the reflection point, $\mathbf{y}$. Moreover, $\mathcal{L}_H(\mathbf{y})$ is the factor that accounts for the contribution of the shape of the reflector to the overall geometrical spreading. For more details, the reader is referred to Schleicher et al. (2001). We also note that $\mathcal{L}_H$ is the reciprocal of the Fresnel relative geometrical spreading introduced in Tygel et al. (1994).

Taking into account the reciprocity of the relative geometrical spreading in equation A-1 and using equation 14 gives

$$\mathcal{L}_R = \mathcal{L}_{NIP}^2\mathcal{L}_H = 2\mathcal{L}_{NIP}\mathcal{L}_N, \tag{A-2}$$

which yields

$$\mathcal{L}_H = \frac{2\mathcal{L}_N}{\mathcal{L}_{NIP}} \, . \tag{A-3}$$

Similarly we have for the KMAH index

$$\chi_R = \chi_{NIP} + \chi_N = 2\chi_{NIP} + \chi_H \tag{A-4}$$

where the KMAH index, χ_H, is related to the phase change at the reflector. It follows that

$$\chi_H = \chi_N - \chi_{NIP} \, . \tag{A-5}$$

CONSTRUCTION OF A HALF-VELOCITY MODEL

In this appendix, we explain how to modify the parameters of a given anisotropic medium so that the velocities in the new medium equal the corresponding ones of the original medium, multiplied by a user user selected constant. In particular, we use this strategy to construct the "half-velocity model" needed for the post-stack migration scheme used in the main text.

We adopt the notation and basics of anisotropic wave propagation from Chapman (2004). We consider a given anisotropic medium specified by the density-normalized matrices

$$\mathbf{a}_{ij} = \mathbf{c}_{ij}/\rho \, , \tag{B-1}$$

where $(\mathbf{c}_{ij})_{kl} = c_{ijkl}$ is the elastic-parameter tensor and ρ is the density function. For a given slowness direction vector, $\hat{\mathbf{p}}$, the associated Christoffel matrix, $\mathbf{\Gamma}$, is defined by

$$\mathbf{\Gamma} = p_i p_j \mathbf{a}_{jk} \, , \tag{B-2}$$

for which the associated eigen-equation is

$$\left(v^2\mathbf{I} - \mathbf{\Gamma}\right)\hat{\mathbf{g}} = \mathbf{0} \, . \tag{B-3}$$

The three eigenvalues, v, (namely, $v = v_I$, with $I = 1,2,3$), are the phase velocities of three different wave modes, which determine the corresponding permitted slowness vectors $\mathbf{p} = \hat{\mathbf{p}}/v$. The corresponding (unit) eigenvectors, $\hat{\mathbf{g}}' = \hat{\mathbf{g}}$, are the polarization vectors.

We consider the equations of motion and constitutive relations of wave propagation in an anisotropic medium, as given in Chapman (2004), equations (4.5.35) and (4.5.36)),

$$\frac{\partial\mathbf{u}}{\partial t} = \frac{1}{\rho}\,\frac{\partial\mathbf{t}_j}{\partial x_j} + \frac{1}{\rho}\,\mathbf{f} \quad \text{and} \quad \frac{\partial\mathbf{t}_j}{\partial t} = \mathbf{c}_{jk}\,\frac{\partial\mathbf{u}}{\partial x_k} \, . \tag{B-4}$$

In the above equations, $\mathbf{u}$, denotes the particle velocity vector and $\mathbf{t}_j$ is the j-th component (column) of the stress tensor. If we divide equations B-4 by a scalar, $K \neq 0$, and then change variables

$$t \to t' = Kt, \quad \rho \to \rho' = K\rho \quad \text{and} \quad \mathbf{c}_{ij} \to \mathbf{c}'_{ij} = \mathbf{c}_{ij}/K \;, \tag{B-5}$$

we see that the new equations are of the same form as the old ones. Moreover, the modified Christoffel equation becomes

$$\left((v')^2\mathbf{I} - \mathbf{\Gamma}'\right)\mathbf{g}' = \mathbf{0} \;, \tag{B-6}$$

or

$$\left((K\,v')^2\mathbf{I} - \mathbf{\Gamma}\right)\mathbf{g}' = \mathbf{0} \;, \tag{B-7}$$

which has the solutions

$$v' = v/K \;, \tag{B-8}$$

and the same polarization vectors, $\mathbf{g}'_I = \mathbf{g}_I$. From the above equations, we have that the new density-reduced elastic tensor, a'_{ijkl}, and the new slowness vector, $\mathbf{p}'$ satisfy

$$a'_{ijkl} = \frac{c'_{ijkl}}{\rho'} = \frac{(c_{ijkl)}/K)}{K\rho} = \frac{1}{K^2}\,a_{ijkl} \tag{B-9}$$

and

$$\mathbf{p}' = \frac{1}{v'}\,\hat{\mathbf{p}} = \frac{1}{(v/K)}\,\hat{\mathbf{p}} = K\mathbf{p} \;. \tag{B-10}$$

In view of the above, the new group velocity, $\mathbf{V}' = d\mathbf{x}/dt$, is given by

$$V'_i = a'_{ijkl}\,p'_k\,\hat{g}_j\,\hat{g}_l = \frac{a_{ijkl}}{K^2}\,(Kp_k)\,\hat{g}_j\,\hat{g}_l \;= V_i/K. \tag{B-11}$$

Equations B-8 and B-11 tell us that the phase and group velocity are both scaled by $1/K$ in the new medium. This is what we expected since the equations of motion and constitutive relations B-4 remained unchanged for the new elastic parameter tensor and new density, together with the fact that the new time has been multiplied by K (see equation B-5). From this we see that, to obtain a half-velocity model, one has to divide the elastic parameter tensor by two and multiply the density by two.