REFLECTIONS FROM A SPHERICAL INTERFACE

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ABSTRACT

Analytical reference traveltimes are convenient for many purposes. Whereas reflection traveltimes for plane interfaces in a homogeneous background can immediately be computed with an exact hyperbolic expression, the situation becomes more complicated as soon as curved reflectors are involved. In this manuscript, I provide an intuitive algorithm for the computation of seismic reflection traveltimes for a spherical reflector in a 3D homogeneous medium.

INTRODUCTION

There are many situations where it is convenient to have access to analytical reference traveltimes for simple media. One example is the verification of software routines, where input data must be reliable.

In this work, I present a solution for the traveltime of a wave reflected from a spherical interface in a homogeneous medium. This is a classic problem that has been addressed in mathematical treatises early on (e.g., Salmon, 1848). I suggest a solution based on Snell's law and geometry that provides more physical insight than the purely mathematical approach.

Since the derivation for the traveltime from Snell's law leads to a polynomial of order six, which cannot be solved analytically, I use results from a related problem, the reflection of a wave from an inclined interface. In combination with these results, I obtain a polynomial of order four for a circular interface, i.e., in 2D, which has an analytical solution. The 3D case can be expressed in terms of the 2D problem. I will, therefore, first discuss reflections from an inclined interface before turning to the problem of the circular, and then, finally, the spherical reflector.

REFLECTIONS FROM AN INCLINED INTERFACE

Figure 1 shows the geometry of the 2D problem considered in this section. A dipping plane interface with the inclination angle ϕ_0 intersects the acquisition surface at the origin denoted 0 with coordinates $\vec{0} = (0,0)$. The raypath from a source denoted S with $\vec{S} = (s,0)$ to a receiver at G with $\vec{G} = (g,0)$ leads to a reflection point at R. The reflection traveltime of the ray $\overline{SRG}$ is $T = \overline{SRG}/V$, where V is the velocity. As can be observed in Figure 1, the length of the ray $\overline{SRG}$ is the same as the distance $\overline{XG}$, where X is the mirror point of S with respect to the reflector.

From the geometry in Figure 1, we immediately see that

$$\overline{SX} = 2\,s\sin\phi_0 \quad , \tag{1}$$

$$\overline{SG} = g - s \quad , \tag{2}$$

and

$$\angle GSX = 90^\circ + \phi_0 \quad \Rightarrow \quad \cos\angle GSX = -\sin\phi_0 \quad . \tag{3}$$

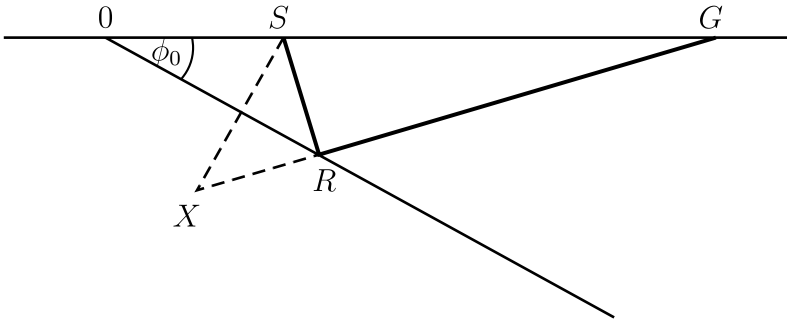

Figure 1: Geometry of a reflection from an inclined interface.

The law of cosines yields

$$\begin{aligned}\overline{XG}^2 &= (g-s)^2 + 4\,s^2 \sin^2\phi_0 + 4\,s(g-s)\sin^2\phi_0 \\ &= (g-s)^2 + 4\,sg\sin^2\phi_0 \\ &= V^2T^2 \quad . \end{aligned} \tag{4}$$

As we will see in the next section, the traveltime from S to R, $T_s = \overline{SR}/V$, as well as the traveltime from R to G, $T_g = \overline{RG}/V$, are also required for the circular and, respectively, spherical reflector problem. I use the law of sines in the triangle SXG, leading to

$$\frac{\sin\angle GSX}{\overline{XG}} = \frac{\sin(90^\circ+\phi_0)}{VT} = \frac{\cos\phi_0}{VT} = \frac{\sin\angle XGS}{\overline{SX}} = \frac{\sin\angle XGS}{2\,s\sin\phi_0} \quad , \tag{5}$$

and with $\angle XGS = \angle RGS$

$$\sin\angle RGS = \frac{2\,\sin\phi_0\cos\phi_0}{VT}\,s \quad . \tag{6}$$

In a similar fashion, I find that

$$\sin\angle GSR = \frac{2\,\sin\phi_0\cos\phi_0}{VT}\,g \quad . \tag{7}$$

The angles $\angle RGS$ and $\angle GSR$ together with the distance $\overline{SG} = g-s$ are sufficient to describe the triangle SRG. With the law of sines in SRG,

$$\frac{\sin\angle RGS}{VT_s} = \frac{\sin\angle GSR}{VT_g} = \frac{\sin\angle GSR}{V(T-T_s)} \tag{8}$$

Finally, I arrive at the following expressions for T_s and T_g in terms of T, s, and g:

$$T_s = \frac{s\,T}{s+g} \quad , \qquad T_g = \frac{g\,T}{s+g} \quad . \tag{9}$$

These expressions are a result that is needed in the following section for the derivation of the traveltime of a reflection from a circular interface.

REFLECTIONS FROM A CIRCULAR INTERFACE

Again, I begin by introducing the geometry of the problem as it is shown in Figure 2. The circle has the radius r. Its centre C is located at $\vec{C} = (0, H)$. The reflection point R for a ray from the source at S to the receiver at G is described by (x_R, z_R), where

$$x_R = r\sin\phi \quad , \qquad z_R = H - r\cos\phi \quad . \tag{10}$$

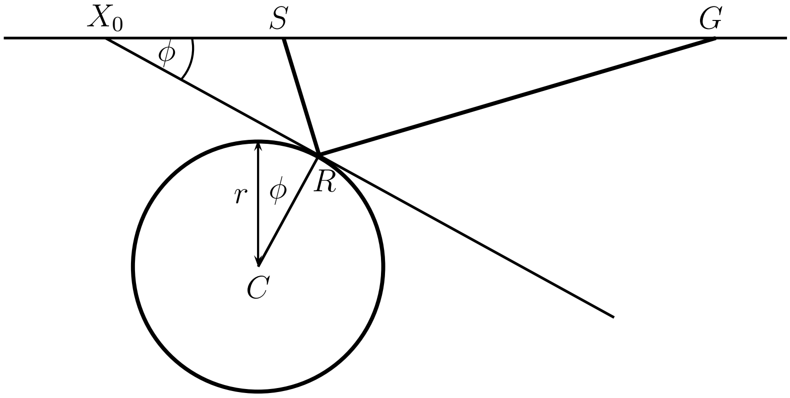

Figure 2: Geometry of a reflection from a circular interface with the radius r: for a fixed reflection point R, the circular reflector can be replaced by an auxiliary plane and inclined reflector that is tangent to the sphere in R. Note that this auxiliary plane intersects the acquisition surface at X_0, unlike the inclined reflector in Figure 1, which intersects the acquisition surface at 0.

Note that the angle ϕ is not known at this point. The traveltimes from S to R, $T_s = \overline{SR}/V$, and that from R to G, $T_g = \overline{RG}/V$, are given by

$$\begin{aligned} V\,T_s &= \sqrt{(s - r\sin\phi)^2 + (H - r\cos\phi)^2} \quad , \\ V\,T_g &= \sqrt{(g - r\sin\phi)^2 + (H - r\cos\phi)^2} \quad . \end{aligned} \tag{11}$$

In order to determine the angle ϕ, I evaluate Snell's law, demanding that $\partial T/\partial\phi = 0$, where $T = T_s + T_g$ is the traveltime of the reflected ray. Carrying out the derivation, I find that

$$\frac{sr\cos\phi - Hr\sin\phi}{T_s} + \frac{gr\cos\phi - Hr\sin\phi}{T_g} = 0 \quad . \tag{12}$$

Substituting the expressions for T_s and T_g, equation (11), into (12) leads to a polynomial of sixth order, which cannot be solved analytically. There is, however, an alternative for using (11) in (12): the reflection from the circle for a fixed reflection point R is equivalent with that from an inclined reflector that is tangent to the circle in R, even though R is not yet known. The inclination angle of this auxiliary reflector is ϕ, which is also not yet known, and it intersects the acquisition plane at X_0 with $\vec{X_0} = (x_0, 0)$, as depicted in Figure 2. In this case, we can use the expressions for T_s and T_g derived in the previous section, Equation (9). Taking the change of the intersection point with the acquisition plane into consideration, I find that

$$T_s = \frac{(s - x_0)\,T}{s + g - 2\,x_0} \quad , \qquad T_g = \frac{(g - x_0)\,T}{s + g - 2\,x_0} \quad . \tag{13}$$

If these are substituted into (12), after some tedious algebra, I obtain the fourth-order polynomial

$$ay^4 + by^3 + cy^2 + dy + e = 0 \quad , \tag{14}$$

where $y = \sin\phi$ and

$$\begin{aligned} a &= -4\,[H^2(s+g)^2 + (H^2 - sg)^2] \\ b &= 4\,r(s+g)(H^2 + sg) \\ c &= 4\,(sg - H^2)^2 - 4\,H^2r^2 - (r^2 - 4\,H^2)(s+g)^2 \\ d &= -4\,rsg(s+g) \\ e &= (r^2 - H^2)(s+g)^2 \quad . \end{aligned} \tag{15}$$

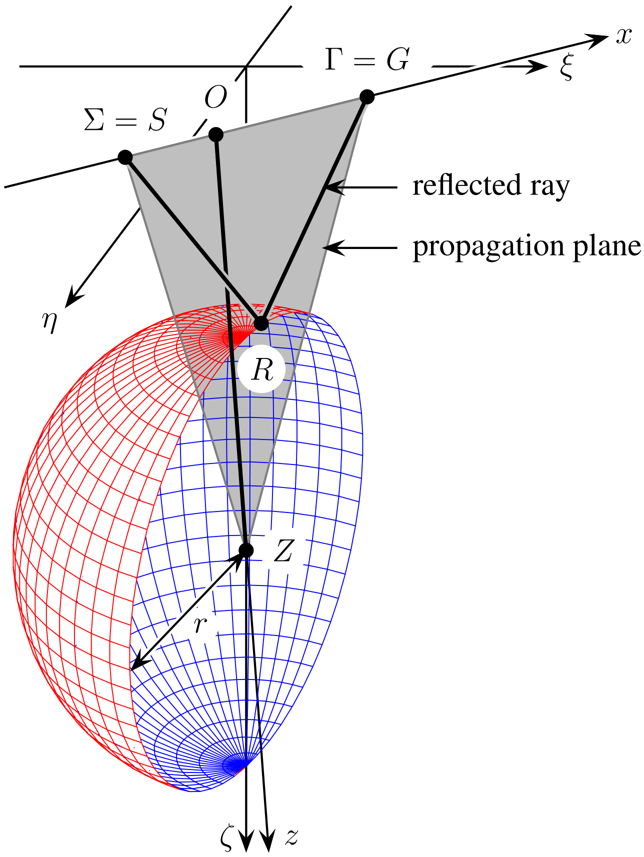

Figure 3: Coordinates and raypaths for the reflection from a sphere and their projections to the 2D equivalent. In the 3D coordinate system defined by (ξ, η, ζ), the source lies at $\vec{\Sigma} = (\xi_s, \eta_s, 0)$ and the receiver at $\vec{\Gamma} = (\xi_g, \eta_g, 0)$. The centre of the sphere is located at $\vec{Z} = (0, 0, \zeta)$. The plane spanned by the source, receiver, and centre of the circle contains also the reflection point R. The corresponding 2D problem with source (S) and receiver (G) aligned along the x-axis with origin at O is obtained from a projection into this plane. The angle made by z and ζ leads to the 2D depth H.

Note that x_0 does not occur in these coefficients. The polynomial (14) has four solutions. Only two of them fulfill Snell's law (12). The reason is that (12) had to be squared in order to obtain (14). Of the two solutions obeying (12), one describes the reflection from the top of the circle, and the other the reflection from the bottom of the circle.

Once the angle ϕ has been determined from (12), T_s and T_g, and therefore the reflection traveltime for the given source-receiver combination can be computed.

In the next section, I will explain the geometrical correspondence between the reflection from a circle in 2D and that from a sphere in 3D, and how the two cases are related.

REFLECTIONS FROM A SPHERICAL INTERFACE

The plane in which a reflected wave travels is generally determined by the plane of the incident wave and the reflector tangent plane, respectively, its normal. Since the normal for a spherical reflector has a radial orientation, this means that wave propagation takes place in the plane defined by the source and receiver position and the centre of the sphere. Since the reflection point R also lies in this plane, the reflection can be treated as a corresponding 2D situation. I will now provide the necessary geometrical relations for the

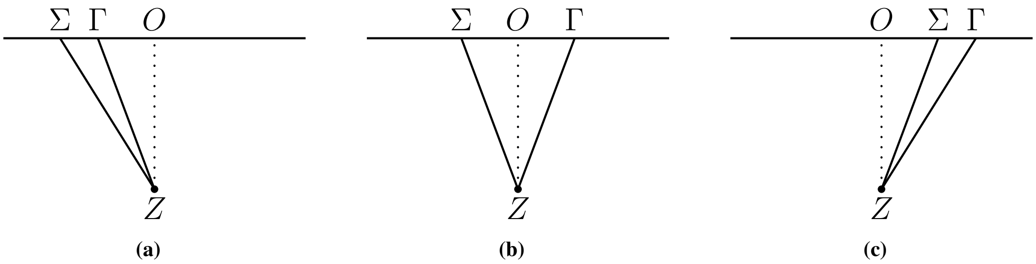

Figure 4: Possible alignments of source (Σ), receiver (Γ), and projection (O) of the centre of the sphere (Z) onto the source-receiver line. Note that these cases comprise those where source and receiver positions are interchanged because reciprocity holds.

application of the results from the previous section.

In order not to confuse the notations in 2D and 3D, Latin letters denote properties in the 2D system, and Greek letters stand for 3D. Accordingly, the source and receiver positions in 3D are Σ at $\vec{\Sigma} = (\xi_s, \eta_s, 0)$ and Γ at $\vec{\Gamma} = (\xi_g, \eta_g, 0)$, respectively, as shown in Figure 3. The centre of the sphere is located at Z with $\vec{Z} = (0, 0, \zeta)$[1].

As I have stated above, the rays contributing to the reflection travel in the plane spanned by Σ, Γ, and Z. To obtain expressions for the 2D equivalents S, G, and C, the (ξ, η, ζ) coordinate system must be rotated to (x, z) coordinates in a fashion that $\vec{S}(\Sigma) = (s, 0)$, $\vec{G}(\Gamma) = (g, 0)$, and $\vec{C}(Z) = (0, H)$.

Note that while the radius of the sphere is the same as that of the circle, the depth H of the centre of the circle is not equal to ζ. This is the case because the depth H is obtained from the distance of Z on the line that connects Σ and Γ. It can be computed by a simple projection of Z on the line that connects Σ and Γ, resulting in the point O, the new origin that corresponds to $(0, 0)$ in the 2D system. In 3D coordinates, the point O is given by

$$\vec{O} = \vec{\Sigma} + \frac{(\vec{Z} - \vec{\Sigma}) \cdot (\vec{\Gamma} - \vec{\Sigma})}{(\vec{\Gamma} - \vec{\Sigma}) \cdot (\vec{\Gamma} - \vec{\Sigma})} (\vec{\Gamma} - \vec{\Sigma}) \quad , \tag{16}$$

and the depth H by the distance between Z and O, i.e.,

$$H = \left| \vec{Z} - \vec{O} \right| \quad . \tag{17}$$

The values for s and g are, accordingly, given by the distances between Σ and O, and Γ and O, respectively. However, in contrast to the depth H, we must now also take into account that the signs of the respective distances may differ. These depend on the positions of Σ and Γ with respect to O, as illustrated in Figure 4.

Any point Π on the line defined by the source and receiver obeys the equation

$$\vec{\Pi} = \vec{\Sigma} + t\,(\vec{\Gamma} - \vec{\Sigma}) \quad . \tag{18}$$

In particular, for $\vec{\Pi} = \vec{\Sigma}$, the parameter t is zero, and for $\vec{\Pi} = \vec{\Gamma}$, t=1. For the three cases depicted in Figure 4, we find that

(a) $t < 0$ and therefore s and g must be chosen as

- $s = -\left|\vec{\Sigma} - \vec{O}\right|$
- $g = -\left|\vec{\Gamma} - \vec{O}\right|$

[1]This choice is due to the lack of a letter in the Greek alphabet that corresponds to 'C'.

(b) $0 < t < 1$ and therefore s and g must be chosen as

- $s = -\left|\vec{\Sigma} - \vec{O}\right|$
- $g = +\left|\vec{\Gamma} - \vec{O}\right|$

(c) $t > 1$ and therefore s and g must be chosen as

- $s = +\left|\vec{\Sigma} - \vec{O}\right|$
- $g = +\left|\vec{\Gamma} - \vec{O}\right|$

With the now determined values for s, g, H, and the radius r, the reflection traveltime between Σ and Γ can be evaluated with the previous results for the reflection from a circle.

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