STABILITY TESTS ON A NUMERICAL SOLUTION OF A PSEUDO-ACOUSTIC TTI WAVE EQUATION

M. Voegele, E. Tessmer, and D. Gajewski

email: *martin.voegele@zmaw.de*
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ABSTRACT

In this work, the performance of two coupled second-order pseudo-acoustic differential equations were tested on various 2D TTI macro-models. At this, we examined particularly the relation between artificially varied amounts of shear-wave velocity and the stability of the modelling. Moreover, the effect of smoothing the underlying macro-model was studied. Contrary to expectations, reducing parameter contrasts may have a positive effect for simple models in a weakly anisotropic environment but turns out to become counter-productive in more complex models or models with stronger anisotropy.
Based on our investigations, we conclude that models with a rapid varying tilt of the symmetry axis and an increased number of interfaces favour the activation of non-physical, with propagation time growing solutions for pseudo-acoustic wave equations using the differential operators presented.

INTRODUCTION

The need for efficient and robust algorithms which take anisotropy into account, leads to a pseudo-acoustic approach which can be derived from the P-SV dispersion relation for the full-elastic system (Alkhalifah, 1998). Unfortunately, modifying this relation by setting the amount of shear-wave velocity along the axis of symmetry artificially to zero not only simplifies qP-wave modelling but also enhances the model results to become susceptible to numerical instabilities in tilted transversely isotropic (TTI) media. This is the case for numerous so far presented pseudo-acoustic wave equations (e.g Zhou et al. (2006); Du et al. (2008); Duveneck et al. (2008)). Especially in acoustic media with a varying tilt of the symmetry axis relative to the coordinate system, all of them turned out to produce unphysical solutions for the wave equation which distort the modelling even after a few iteration steps (Bube et al., 2012). Coming from the conjecture that zero shear velocities normal to the symmetry plane cause numerical instability, Fletcher et al. (2009) advertised a new finite shear-wave velocity attempt which turned out to become numerically unstable as well in TTI media.
Based on their approach, we examine the performance of the two coupled second-order differential equations (PDEs). The main purpose of these studies is to get a better idea of how to stabilise and (at best) prevent the activation of non-physical solutions of the wave equation in a TTI environment.

DERIVATION OF THE PSEUDO-ACOUSTIC WAVE EQUATION

The pseudo-acoustic wave equation is based on the dispersion relation of the three wave modes P, SV and SH which are obtained when solving the Christoffel equation for its eigenvalues and associated eigenvectors.

Since the SH-mode is decoupled, the P-SV dispersion relation for the phase velocities in the general 3D case is given by

$$\begin{aligned}\omega^4 = {} & [(V_{px}^2 + V_{sz}^2)(\hat{k}_x^2 + \hat{k}_y^2) + (V_{pz}^2 + V_{sz}^2)\hat{k}_z^2]\omega^2 - V_{px}^2 V_{sz}^2(\hat{k}_x^2 + \hat{k}_y^2)^2 \\ & - V_{pz}^2 V_{sz}^2 \hat{k}_z^4 + [V_{pz}^2(V_{pn}^2 - V_{px}^2) - V_{sz}^2(V_{pn}^2 - V_{pz}^2)] \cdot (\hat{k}_x^2 + \hat{k}_y^2)k_z^2 \,,\end{aligned} \tag{1}$$

where ω is the angular frequency, V_{pz} and V_{sz} are the P and S-wave velocities parallel to the symmetry axis and $V_{pn} = V_{pz}\sqrt{1+2\delta}$ is defined as the normal moveout velocity (Fletcher et al., 2009). The quantity $V_{px} = V_{pz}\sqrt{1+2\varepsilon}$ denotes the P-wave velocity parallel to the symmetry plane. By setting $V_{sz} = 0$, the dispersion relation corresponds to the approximation from which the first pseudo-acoustic wave equation was derived by Alkhalifah (1998). The parameters $\hat{k}_x, \hat{k}_y$ and $\hat{k}_z$ are rotated wavenumbers with respect to the symmetry axis in TI media.

The solution for equation 1 is given by one fourth order or two coupled second-order partial differential equations in time that are obtained by substituting the expression for the wave numbers from equation 4 into the dispersion relation 1 and multiplying both sides with the pressure wave field $p(\omega, k_x, k_y, k_z)$. Then, an inverse Fourier transform to both sides leads to the final equations presented by Fletcher et al. (2009)

$$\begin{aligned}\frac{\partial^2 p}{\partial t^2} &= V_{px}^2 H_2 p + \alpha V_{pz}^2 H_1 q + V_{sz}^2 H_1(p - \alpha q) + S \\ \frac{\partial^2 q}{\partial t^2} &= \frac{V_{pn}^2}{\alpha} H_2 p + V_{pz}^2 H_1 q - V_{sz}^2 H_2\left(\frac{1}{\alpha}p - q\right) + S \,.\end{aligned} \tag{2}$$

Here, $q(\omega, k_x, k_y, k_z)$ denotes an auxiliary wave field and $\alpha = 1$ is a non-zero scaling factor. In the degenerated isotropic case, wave field $p(\omega, k_x, k_y, k_z)$ equals $q(\omega, k_x, k_y, k_z)$ and both PDEs from 2 are identical. The variable S represents the source function and H_1 and H_2 are two introduced differential operators that contain single and mixed-space derivatives (for more details see appendix A).

Since the numerical studies are performed in the two-dimensional x-z-domain, all derivatives with respect to the y-direction are zero. For the sake of minimising computational resources, spatial derivatives in equation 2 are solved using a pseudo-spectral algorithm introduced by Kosloff and Baysal (1982). The time integration is approximated using fourth-order Taylor expansions.

NUMERICAL STUDIES

The stability of the wave equation has been studied exemplary for the two-dimensional case on 105 synthetic anisotropic models ranging from homogeneous VTI models to more complex geological structures with strong anisotropy and a rapid varying tilt angle θ. The reason for this large number of experiments is to find a common thread in

(a) the parametrisation of the anisotropy

(b) the complexity of the geology

of the models that are likely to end in numerical instability.

We tested each model with 14 different shear-wave velocities, in order to point out their effect on the modelling. In addition to that, we investigated whether smoothing the underlying macro-model can be used to preserve stability. For this report, three subsurface models have been chosen representatively of which the studies will be demonstrated in detail.

Table 1: Parameters of representative macro-models

	ϵ	δ	θ
Model 1	0.1	0.01	60°
Model 2	0.11	-0.035	40°
Model 3	0.15	0.081	0°, 31°, 51°, 60°

Model 1 and model 2 are two-layered models with an isotropic top layer with V_{pz} = 3000 m s^{-1} and a subjacent weakly TTI layer. The lower layer of model 2 resembles a Taylor sandstone where the parameters are taken from Thomsen (1986). Model 3 represents a geologically more complex anisotropic structure with an overhang in an isotropic environment. It is based on a model from Fletcher et al. (2009) and is characterised by velocity contrasts between 2740 and 6000 m s^{-1} and strong variations in the tilt axis between 0° and 60°.
Apart from the Thomsen parameters that define the anisotropy in the models, a fourth parameter σ is introduced to describe the kinematics of SV-waves (e.g. Tsvankin, 2001):

$$\sigma \equiv \left(\frac{V_{pz}}{V_{sz}}\right)^2 (\varepsilon - \delta). \tag{3}$$

The value of σ has been chosen in the range of 0.2 - 6 for each model. For $\sigma \to \infty$, the SV-velocities in the model become zero. Unfortunately, strong diamond-shaped shear-wave artefacts may impair the image and the ratio V_{pz}/V_{sz} becomes physically unrealistic (Fletcher et al., 2009). For $\sigma \to 0$, triplications in the SV-wave front disappear in strong anisotropic media. However, in this case the shear-wave velocities may become unphysically large.

RESULTS

General stability issue

Presenting the global maximum pressure amplitude over time for the example cases illustrates the well-known stability issue of the pseudo-acoustic wave equation.
Figure 1a) shows the global pressure trend of a wave field propagating in the weakly anisotropic two-layered model 1 with varying shear-wave velocities governed by σ. Despite a strong vertical variation of the tilt angle θ at the boundary layer, wave propagation can be considered as stable over time.
Figure 1b) presents the trend of the global maximum pressure for model 2 in a weakly anisotropic environment of a Taylor sandstone. At first glance, the result for the propagating wave field remains relatively constant for all values of σ. However, after roughly 12000 iteration steps, the pressure amplitudes of all σ values except σ = 0.2 start to grow.
In contrast, the pressure trend of the geologically more complex model 3 is presented in figure 1c). Here, the pressure amplitudes of the propagating wave fields become unstable after roughly 3100 iteration steps. At this point, wave fields with smaller shear-wave velocities tend to become unstable more easily. By comparing the gradient of the amplitude trends, one can recognise a slightly steeper slope for larger values of σ.

Combined with findings obtained from other test runs, the connection between parametrisation, number of boundaries in the geology and unstable model results over time turn out to be very complex. Therefore, a direct determination of whether a subsurface configuration is suitable for modelling or not is impossible. Consequently the performance of smoothing is tested to reduce parameter contrasts in the model and delay instabilities occurring during modelling.

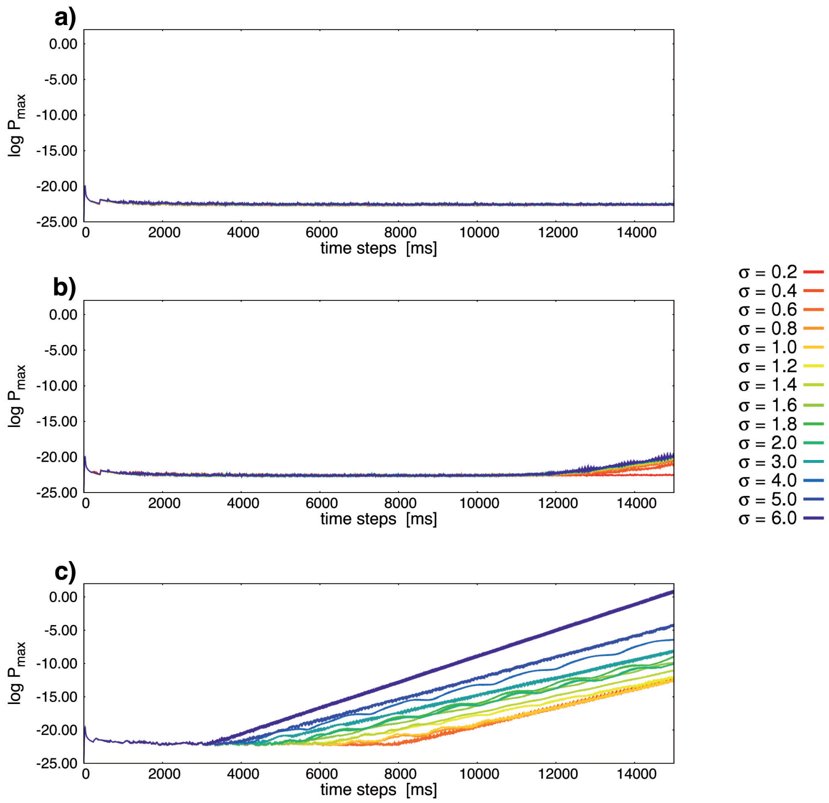

Figure 1: Global maximum pressure over time for (a) model 1 with a weakly anisotropic layer, (b) model 2 with a subjacent weakly anisotropic Taylor sandstone layer and (c) model 3 with a variable tilt angle for varying values of σ. The amplitudes are scaled logarithmically.

Smoothed macro-models

Fletcher et al. (2009) hints at rapid alterations of the orientation of the symmetry plane as the origin of numerical instabilities in the model. For the numerical studies, the macro-models were smoothed iteratively with a weighted two-dimensional window of second order. The intensity of smoothing has been controlled by applying the smoothing operator repeatedly to the model. For smoothing seismic velocities, the filter is applied on their slowness in order to preserve time-depth relations. The tilt angle distribution is smoothed without inversion. The result leads to smaller amplitude contrasts at the cost of decreasing spatial resolution of the structures in the model. Fortunately, not too strong smoothing the macro-model does not necessarily lead to inaccurate migration results using pre-stack RTM (Tessmer, 2003).

Figure 2 illustrates the results of applying the smoothing operator a) zero times, b) 500 times and c) 1000 times on the weakly anisotropic macro-model 2 which will serve as an example in this section. Smoothing introduces spatial parameter changes to the macro-model. As a result, the formerly stable numerical computations for a wave field in model 1 becomes highly unstable in figure 2b) 500 smoothing iterations and figure c) 1000 iterations. Whereas the activation of exponentially growing solutions for the PDEs arise shortly after 4000 time steps in the former case, doubling the application of the smoothing operator not only reduces the contrasts in the macro-model stronger, but also delays exponentially growing pressure amplitudes. Again, higher values of σ advance the activation of numerical instabilities.

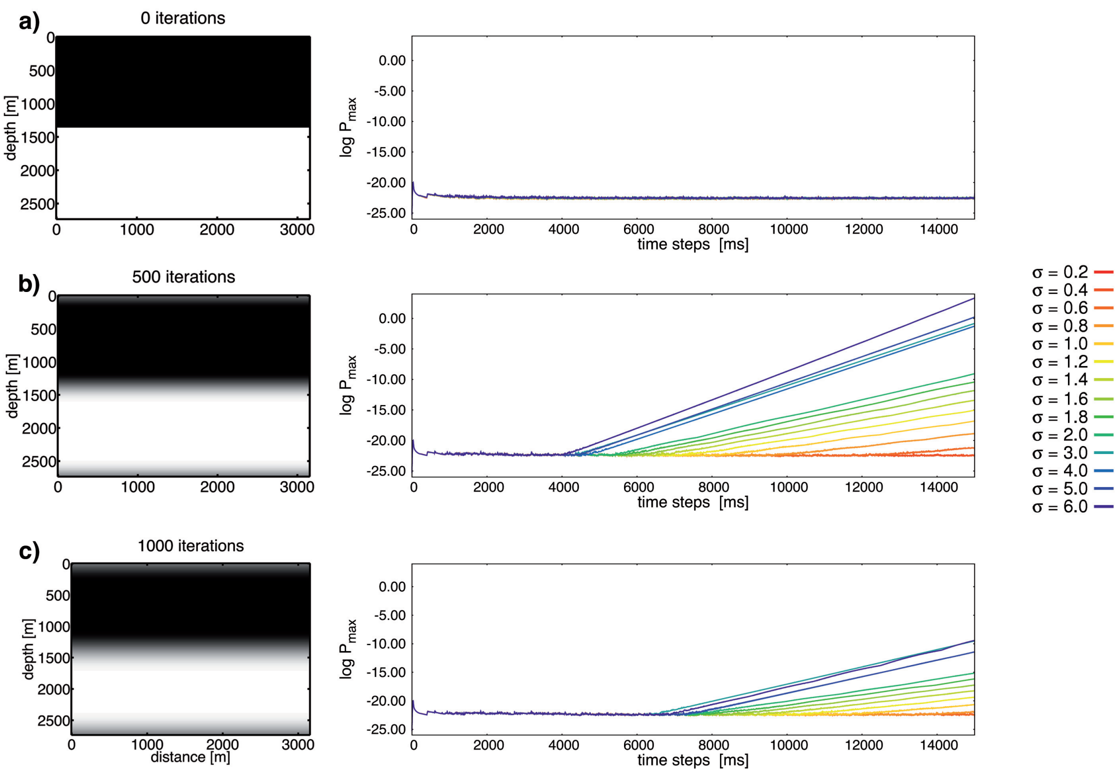

Figure 2: Outcome of smoothing the weakly anisotropic model 1 with a different number of iterations: (a) no smoothing, (b) 500 smoothing iterations and (c) 1000 smoothing iterations. The amplitudes are scaled logarithmically.

CONCLUSIONS

In this report a pseudo-acoustic wave equation was tested in TTI media. The first part of the numerical studies considered the performance of the algorithm in 105 two-dimensional macro-models that have been defined through anisotropic parameters, the tilt angle of the symmetry axis, the vertical P-wave velocity and their geometries. Each model has hereby been tested with 14 different shear-wave velocities which are controlled by the parameter σ.
Since each parameter value and its associated distribution alters the model results, the formerly stability issue which has been linked with zero shear-wave velocities turns out to be associated additionally with the amount of anisotropy and the number of interfaces in the model. Consequently, the complexity of the model geometries was limited to the homogeneous, two-layered and multi-layered case in order to facilitate the interpretation of the results.
By trying to find a common thread within the parametrisation of the models, their geometry and the respective robustness of the model solutions, it is striking that increasing the number of interfaces within a model while the anisotropic parameters are kept constant advances the activation of numerical instabilities. Vice versa, the degree of anisotropy amplifies occurring non-physical solutions of the PDE. Larger values for σ lead to a stronger activation of instabilities.

Fletcher et al. (2009) stated in their work, that rapid variations in the tilt angle of the symmetry axis favour numerical instability. This is in accordance with our findings since smoothing the model (and thus increasing the number of interfaces and parameter variations in the model) impairs the computations tremendously. In the case of TTI models with a high number of variations in the tilt axis and moderate to strong anisotropy, the pressure amplitudes are likely to end in floating-point exceptions within few time steps. Coming from the conjecture that an increased number of interfaces in the model compromises the robustness of the algorithm, it is understandable that smoothing is counter-productive.

Contrary to the conclusions of Fletcher et al. (2009), our studies reveal that even for non-zero shear-wave velocities, long propagation times lead to instability. As opposed to Bube et al. (2012) and Fletcher et al. (2009) who suggested that zero shear-wave velocities are the reason for numerical instabilities, we suggest that time-growing solutions for the wave equation are likely to be inherent to the PDEs and result from the implementation of the rotated differential operators (Zhang et al., 2011) (see appendix A).

PUBLICATIONS

More information on this work is presented in the Bachelor's thesis of Voegele (2013).

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APPENDIX A

ADDITIONAL INFORMATION ON THE PSEUDO-ACOUSTIC WAVE EQUATION

Since the P-SV dispersion relation 1 takes TTI media into account, two angles θ and ϕ are adopted in the definition of the wave numbers $\hat{k}_x$, $\hat{k}_y$ and $\hat{k}_z$ in the rotated system.

$$\begin{aligned}
\hat{k_x} &= k_x \cos\theta\cos\phi + k_y \cos\theta\sin\phi - k_z \sin\theta \ , \\
\hat{k_y} &= -k_z \sin\phi + k_y \cos\phi, \\
\hat{k_z} &= k_x \sin\theta\cos\phi + k_y \sin\theta\sin\phi + k_z \cos\theta
\end{aligned} \tag{4}$$

Here, θ is the tilt angle of the symmetry plane relative to the vertical axis of the coordinates and ϕ is the azimuth of the symmetry axis.

The two introduced differential operators H_1 and H_2 are given by:

$$\begin{aligned}
&H_1 = \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial x^2} + \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial y^2} + \cos^2\theta\frac{\partial^2}{\partial z^2} + \sin^2\theta\sin 2\phi\frac{\partial^2}{\partial x\partial z} \\
&+ \sin 2\theta\sin\phi\frac{\partial^2}{\partial x\partial y} + \sin 2\theta\cos\phi\frac{\partial^2}{\partial x\partial y} \ , \\
&H_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - H_1 \ .
\end{aligned} \tag{5}$$